Domination Polynomial: The Study of Various Graph Operations

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ABSTRACT

Let

$G = V, E$ ***be a graph of order n. The independent domination polynomial of G is the polynomial***

$$D_i\ G, X = \sum_{j=\gamma_i(G)}^{n} d_i\ G, j\ x^j,$$

where is the quantity of free overwhelming arrangements of size . In this paper, we present the free mastery polynomial of a chart. The autonomous mastery polynomials of some standard diagrams are acquired and a few properties of the free control polynomial of charts are set up

Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,\lambda) = \sum_{i=0}^{n} d(G,i)\lambda^i$, where $d(G,i)$ is the number of dominating sets of G of size i. Every root of $D(G,\lambda)$ is called the domination root of G. In this paper, we study the domination polynomial of some graph operations.

Keywords: Dominating polynomial, independent dominating polynomial, independent dominating polynomial roots.

I INTRODUCTION

An associated ruling arrangement of a diagram G is a set D of vertices with two properties:

(a) Any hub in D can achieve some other hub in D by a way that stays entirely inside D. That is, D prompts an associated subgraph of G

(b) Every vertex in G either has a place with D or is contiguous a vertex in D.

That is, D is an overwhelming arrangement of G. A base associated commanding arrangement of a diagram G is an associated ruling set with the littlest conceivable cardinality among all associated vast arrangements of G. The associated control number of G is the number of vertices in the base associated ruling set. By the meaning of associated mastery number, (G) is the base cardinality of an associated ruling set in G. For more insights about control number and its related parameters, we allude to [1] – [4]. For a point by point treatment of the mastery polynomial of a diagram, the reader is alluded to [5], [6]. We present the associated mastery polynomial of G, we get associated control polynomial and register its underlying foundations for some standard diagrams.

In a graph G V, E, the open neighborhood of a vertex v V(G) is N v x V; vx E the arrangement of vertices nearby v. The shut proximity is N v N v {v}. The subgraph actuated by the set S. A set S V is an overwhelming set if each vertex in V – S is nearby a vertex of S and the base cardinality of a vast set is known as the mastery number of G and is indicated by (G). A base overwhelming of a chart G is known as a - set of G.

2.1 Definition

Let G be a simple connected graph. The connected domination polynomial of G is defined by $C_d(G,x) = \sum_{i=\gamma_d(G)}^{|V(G)|} c_d(G,i)\, x^i$, where $\gamma_d(G)$ is the connected domination number of G

2.2 Theorem

Let G be a graph with $|V(G)| = n$. Then

(i) If G is connected then $C_d(G,n) = 1$ and $C_d(G,n-1) = n$

(ii) $C_d(G,i) = 0$ if and only if $i < \gamma_d(G)$ and $i > n$.

(iii) $C_d(G,x)$ has no constant and first degree terms

(iv) $C_d(G,x)$ is a strictly increasing function in $[0,\infty)$.

(v) Let G be a graph and H be any induced subgraph of G. Then $\deg(C_d(G,x)) \geq \deg(C_d(H,x))$.

(vi) Zero is a root of $C_d(G,x)$ with multiplicity $\gamma_d(G)$.

Proof:

(i) Since G has n vertices, there is only one way to choose all these vertices and it connected and dominates all the vertices. Therefore, $c_d(G,n) = 1$. If we delete one vertex v, the remaining n-1 vertices are connected dominate all the vertices of G. (This is done in n ways). Therefore, $c_d(G,n-1) = n$.

(ii) Since $\mathcal{C}_d(G,i) = \Phi$ if $i < \gamma_d(G)$ or $\mathcal{C}_d(G,n+k) = \Phi$, k = 1,2,... Therefore, we have $c_d(G,i) = 0$ if $i < \gamma_d(G)$ or $i > n$. Conversely, if $i < \gamma_d(G)$ or $i > n$, $c_d(G,i) = 0$. Hence the result

Theorem 2.1. *For every $n \in \mathbb{N}$,*

$$D(F_n, x) = (2x + x^2)^n + x(1 + x)^{2n}.$$

Proof. The join $G = G_1 + G_2$ of two graph G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. An elementary observation is that if G_1 and G_2 are graphs of orders n_1 and n_2, respectively, then

$$D(G_1 \cup G_2, x) = D(G_1, x)D(G_2, x)$$

and

$$D(G_1 + G_2, x) = \left((1 + x)^{n_1} - 1\right)\left((1 + x)^{n_2} - 1\right) + D(G_1, x) + D(G_2, x).$$

Clearly $D(K_1, x) = x$ and $D(K_2, x) = 2x + x^2$, so by the previous observations,

$$\begin{aligned} D(F_n, x) &= D(K_1 + nK_2, x) \\ &= (1 + x - 1)^1((1 + x)^{2n} - 1) + x + (2x + x^2)^n \\ &= (2x + x^2)^n + x(1 + x)^{2n}. \quad \square \end{aligned}$$

3.1 Theorem

If F_m is a friendship graph with 2m+1 vertices, then the connected domination polynomial of F_m is $C_d(F_m, x) = x\,[(1+x)^{2m} - 1]$ and the connected dominating roots are 0 with multiplicity 2 and $e^{\frac{i\pi}{m}} - 1$, $e^{\frac{i2\pi}{m}} - 1, \ldots, e^{\frac{i(m-1)\pi}{m}} - 1$ with multiplicity 1.

Proof:

Let G be a friendship graph of size 2m + 1 and m ≥ 2. By labeling the vertices of G as $v_1, v_2, \ldots, v_{2m+1}$ where v_1 is joined with all the vertices and $(v_2,v_3), (v_4,v_5), \ldots, (v_{2m}, v_{2m+1})$ are joined itself. Clearly there are 2m connected dominating set of size two namely $\{v_1, v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_{2m+1}\}$. Similarly for the connected dominating set of size three, we need to select the vertex v_1 and two vertices from the set of vertices $\{v_2, v_3, \ldots, v_{2m+1}\}$. That means there are $\binom{2m}{2}$ connected dominating sets. In general, $c_d(G,i) = \binom{2m}{i-1}$, $2 \le i \le 2n+1$.

$$\begin{aligned} \text{Hence } C_d(F_m, x) &= 2m\,x^2 + \binom{2m}{2}x^3 + \ldots + \binom{2m}{2m}x^{2m+1} \\ &= x\,[(1+x)^{2m} - 1]. \end{aligned}$$

Consider, $x\,[(1+x)^{2m} - 1] = 0$. The roots of this polynomial are 0 with multiplicity 2 and $e^{\frac{i\pi}{m}} - 1$, $e^{\frac{i2\pi}{m}} - 1, \ldots, e^{\frac{i(m-1)\pi}{m}} - 1$ with multiplicity 1.

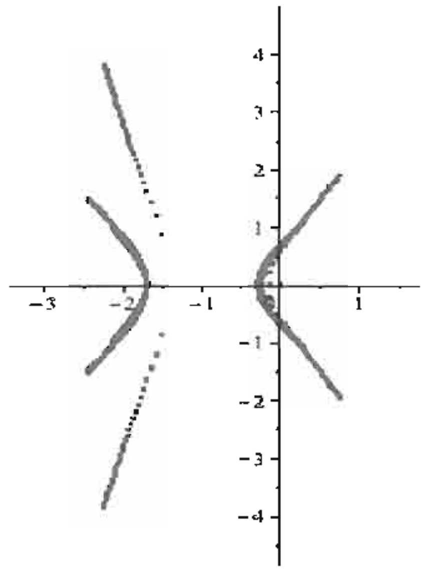

Figure 2: Domination roots of graphs F_n, for $1 \leq n \leq 30$.

II MAIN RESULTS

Just like the case with other diagram polynomials, for example, chromatic polynomials and autonomy polynomials, it is normal to consider the mastery polynomial of a piece of two diagrams. It isn't difficult to see that the equation for control polynomial of the join of two diagrams is acquired as takes after. This task of charts is commutative. Utilizing this item, one can develop an associated diagram *G* with the quantity of commanding sets *n*, where *n* is a subjective odd normal number; see [5]. Let to consider the crown of two charts. The accompanying hypothesis gives us the control polynomial of charts of the shape $H \circ K$ 1 which is the primary outcome for mastery polynomial of particular crown of two charts.

Theorem 1 (see [1]). *Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively. Then*

$$D(G_1 + G_2, x) = ((1+x)^{n_1} - 1)((1+x)^{n_2} - 1) + D(G_1, x) + D(G_2, x). \quad (1)$$

Theorem 11. *(i) Suppose that $G_{n,t} = P_n \diamond K_t$. Then*

$$D(G_{n,t}, x) = \left((x+1)^t - 1\right) \times \left[D(G_{n-1,t}, x) + D(G_{n-2,t}, x) + D(G_{n-3,t}, x)\right]. \quad (8)$$

(ii) Suppose that $H_{n,t} = C_n \diamond K_t$. Then

$$D(H_{n,t}, x) = \left((x+1)^t - 1\right) \times \left[D(H_{n-1,t}, x) + D(H_{n-2,t}, x) + D(H_{n-3,t}, x)\right]. \quad (9)$$

(iii)

$$D(G_3^n \diamond K_t, x) = \left((1+x)^{2t} - 1\right)^n + \left((1+x)^t - 1\right)(1+x)^{2nt}. \quad (10)$$

Proof. (i) From Theorem 8, we have $D(G_{n,t}, x) = D(P_n, (1+x)^t - 1)$. Now by Part (i) of Theorem 9, we have the result.

(ii) From Theorem 8, we have $D(H_{n,t}, x) = D(C_n, (1+x)^t - 1)$. Now by Part (ii) of Theorem 9, we have the result.

(iii) From Theorem 8, we have $D(G_3^n \diamond K_t, x) = D(G_3^n, (1+x)^t - 1)$. Now by Theorem 10, we have the result. □

Definition 2.1. Let $G = V. E$ be a graph of order n with independent domination number the $\gamma_i(G)$ the independent domination polynomial of G is $D_i\ G. X = \sum_{j=\gamma_i(G)}^{n} d_i\ G. j\ x^j$, *where* $d_i\ G. j$ is the number of independent dominating sets of size j. The roots of the polynomial $D_i\ G. X$ are called the independent dominating roots of G.

Example. Let G = (V, E) be a graph as in Figure 1.

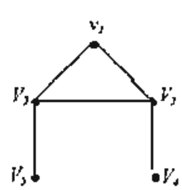

Figure 1

$$D_i(G, x) = \prod_{j=1}^{k} D_i(G_j, x)$$

Now we have to prove that the result is true for n = k + 1,

So let $G \cong \bigcup_{j=1}^{k+1} G_j \cong \bigcup_{j=1}^{k} G_j \cup G_{k+1}$.

$$D_i(G,x) = D_i\left(\bigcup_{j=1}^{k} G_j \cup G_{k-1}\right)$$

$$= D_i\left(\bigcup_{j=1}^{k} G_j, x\right) D_i(G_{k+1},x) \text{ by Theorem 2.3}$$

$$= \prod_{j=1}^{k} D_i(G_j,x).D_i(G_{k+1},x)$$

$$= \prod_{j=1}^{k+1} D_i(G_j,x).$$

Hence $D_i(G, x) = \prod_{j=1}^{n} D_i(G_j,x)$.

Clearly $\gamma_i(G) = 2$ and there are only two minimum independent dominating sets $\{v_3, v_5\}$, $\{v_2, v_4\}$ and one independent dominating set $\{v_1, v_4, v_5\}$ of size 3. Hence, $D_i(G, x) = x^2(2+ x)$

Obviously there are two independent dominating roots of G which are 0 and -2.

Observation 2.2. For any graph G = (V, E) The independent Dominating polynomial of G is $D_i(G,x) = \sum_{j=\gamma_i(G)}^{\beta(G)} d_i(G,j)\, x^j$ where $\beta(G)$ is the maximum independent number and $\gamma_i(G.J)$, is the number of independent dominating set of size j.

Theorem 2.3. Let $G = G_1 \cup G_2$. Then $D_i(G_j,x) = D_i(G_1,x)\, D_i(G_2,x)$.

Proof. Any independent dominating set of k vertices in G is constructed by choosing an independent dominating set of j vertices in G_1 (for some $j \in \{\gamma_i(G_1), \gamma_i(G_2), \ldots |V(G_1)|\}$ and the independent dominating set of k – j vertices in G_2. The number of ways of doing this overall j = $\gamma_i(G_1), \ldots, |V(G_1)|\}$ is exactly the coefficient of x^k in $D_i(G_1,x)\, D_i(G_2,x)$.

Theorem 2.4. Let $G \cong \bigcup_{j=1}^{n} G_j$. Then = $D_i(G, x) = \prod_{j=1}^{n} D_i(G_j, x)$.

Proof. We prove this by mathematical induction, the result is true for j=1 is trivial and by Theorem (2.2) for j = 2.

Suppose that $D_i(G, x) = \prod_{j=1}^{n} D_i(G_j, x)$ is satisfy for n=k

i.e., For G = $\bigcup_{j=1}^{k} G_j$

Theorem 2.11. Let $K_{1,m}$ be a star and G = (V,E) be the spider graph which constructed by subdivision $K_{1,m}$ where m ≥ 3. Then

$$D_i(G,x) = (2^m-1)x^m + x^{m+1}.$$

Proof. Let G = (V, E) be the spider graph, which we get from $K_{1,m}$ by subdivision as in figure 2.

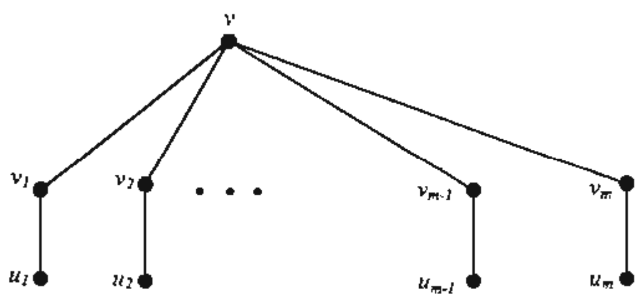

Figure 2. *Healthy spider of* $2n+1$ *vertices*

Observing $\gamma_i(G) = \gamma(G) = m$. Let $A = \{v_1, v_2, v_3, \ldots, v_{m-1}, v_m\}$ and $B = \{u_1, u_2, u_3, \ldots, u_{m-1}, u_m\}$ .To find the number of minimum independent dominating sets which of the size m. We can take one vertex from A and m-1 vertices from B, two vertex from A and m-2 vertices from B and so on.

i,e, $d_i(G, m) = \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m-1} + \binom{m}{m} = \sum_{i=0}^{m} \binom{m}{i} - 1 = 2^m - 1$

Also, there is only one independent dominating set of size m+1, which is $\{v, u_1, u_2, \ldots v_m\}$, d_i (G, m + 1) = 1. Hence D_i (G, x) = x^m (2^m-1+x).

Theorem 2.8. [33] *Let G be a graph and $u \in V$. Then*

$$D(G,x) = D(G-u,x) + D(G \odot u, x) - D(G \odot u - u, x).$$

We are now ready to give a formula for the domination polynomial of B_n.

Theorem 2.9. *For every $n \in \mathbb{N}$,*

$$D(B_n,x) = (x^2+2x)^n(2x+1) + x^2(x+1)^{2n} - 2x^n.$$

Proof. Consider graph B_n and vertex v in the common edge (see Figure 4). By Theorem 2.7 we have:

$$\begin{aligned} D(B_n,x) &= xD(B_n/v,x) + D(B_n-v,x) + xD(B_n - N[v],x) - (1+x)p_v(B_n,x) \\ &= xD(B_n/v,x) + D(B_n-v,x) + x(D(nK_1,x)) - (1+x)x^n \\ &= xD(B_n/v,x) + D(B_n-v,x) - x^n. \end{aligned}$$

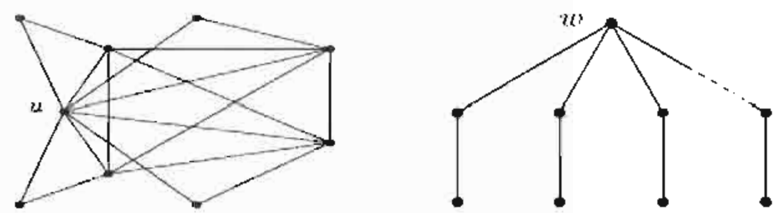

Figure 5: Graphs B_4/v and $B_4 - v$, respectively.

Now we use Theorem 2.8 to obtain the domination polynomial of the graph B_n/v (see Figure 5). We have

$$D(B_n/v,x) = D((B_n/v)-u,x) + D((B_n/v)\odot u,x) - D(2nK_1,x),$$

where $(B_n/v) - u = K_n \circ K_1$ and $(B_n/v) \odot u = K_{1,2n}$ (see Figure 6).

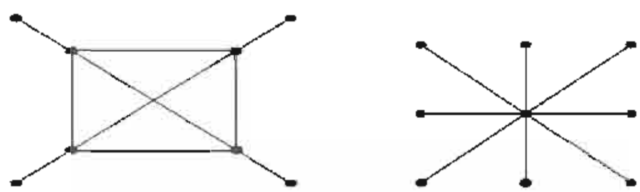

Figure 6: Graphs $B_4/v - u$ and $B_4/v \odot u$, respectively.

Using Theorem 2.8, we deduce that, $D(B_n/v,x) = (2x+x^2)^n + x(x+1)^{2n}$. Also we use Theorems 2.6 and 2.7 to obtain the domination polynomial of the graph $B_n - v$ (see Figure 5). Hence $D(B_n - v,x) = xD((B_n-v)/w,x) + D(K_2,x)^n - x^n$, where $(B_n - v)/w = K_n \circ K_1$. So $D(B_n - v,x) = (2x+x^2)^n(x+1) - x^n$. Note that in this case $p_v(B_n,x) = p_w(B_n - v,x) = x^n$. Consequently,

$$\begin{aligned} D(B_n,x) &= x((2x+x^2)^n + x(x+1)^{2n}) + (2x+x^2)^n(x+1) - x^n - x^n \\ &= (x^2+2x)^n(2x+1) + x^2(x+1)^{2n} - 2x^n. \quad \square \end{aligned}$$

III CONCLUSION

The free ruling polynomial of a chart is one of the logarithmic portrayals of the diagram and nature of any diagram portrayal depend about what data would we be able to get from that introduction about the diagram. As this paper is presented the idea of autonomous ruling polynomial a still there are a considerable measure of issues can be unraveled later on about this idea case isn't for constrained, the examination of the underlying foundations of the free commanding polynomial and order of the diagrams we can set the accompanying open issues: (1) Arrangement of charts which has genuine free ruling roots. (2) Characterization of pictures which has just three unmistakable roots. (3) The summation of the underlying foundations of free control polynomial of a graph gives us another parameter of a chart As of late we chip away at all these open issues, and it will show up as of late.

(i) What is the fundamental recipe for the control polynomial of the Cartesian result of two charts? For two graphs*G*and*H*, let [*H*] be the chart with vertex set *V*(*G*) × *V*(*H*) and to such an extent that vertex (*a*, *x*) is nearby

vertex (b, y) if and just if a is nearby b (in G) or$a = b$and x is nearby y (in$H$). The chart $[H]$ is the lexicographic item (or arrangement) of G and H and can be thought of as the chart emerging from G and H by substituting a duplicate of H for each vertex of G. There is a primary issue.

(ii) By what method can register the control polynomial of Lexicographic result of two diagrams?

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