$M^X/G/1$ RETRIAL QUEUE WITH TWO PHASE REPAIR

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Abstract

In this study, we provide the analysis of a single unreliable server batch arrival queue with repair in two phases. The arrival rate of jobs at service station varies according to the server's status as server being idle, busy, under setup and under i^{th} $(i=1,2)$ phase repair. We assume that the jobs which find the server busy or broken down and under repair, leave the service station and join the retrial orbit with some probability. The service time taken by the server is general distributed. The server always has the tendency to breakdown and the repairman restores the broken down server in two phases. The life time of server is exponentially distributed but the repair in each phase is performed according to general distribution. The queueing analysis has been done using supplementary variable technique and generating function method. Various performance indices are derived in explicit form which can be computed easily as results depend only upon the first two moments of inter-failure times, phase repair times and batch size distributions. The cost analysis has been attempted successfully. Some special cases of interest have been deduced by setting appropriate parameter values. To show the effect of parameters upon various performance characteristics of the system such as mean queue length, probability that the server is idle, busy, under set up, and under phase repair, etc., the sensitivity analysis is carried out by taking numerical illustration.

Keywords:- $M^X/G/1$, Retrial queue, Set up, Unreliable server, Two phase repair, Supplementary variable, Queue size, Reliability

I. INTRODUCTION

All manufacturing environments operate under uncertain conditions stemming from uncertainty about processing time of jobs or the reliability of resources, such as machines/servers. The system designer may recommend the proper repair facility for a system wherein an unreliable server is busy in processing various jobs. The study of a queueing model with unreliable server is of relevance to engineering systems as it helps in decision-making. For the machine repair case, the engineering management decisions are required regarding the type and number of machines, number of repair crew, regular maintenance versus repair-or-breakdown philosophies, etc.. Queueing models are needed to have an insight for congestion related issues such as waiting times, idle times, busy times and many other system characteristics.

Retrial queueing systems are characterized by the feature that the arriving jobs, who on finding the server busy join the retrial group/orbit to try again after random intervals for their requests; the reason being that the return of customers plays a special role in many of these systems as well as in other practical applications of these systems. The retrial has a non-negligible negative effect on the performance measures. It also has wide applicability in real world congestion situations such as manufacturing systems, production systems, distribution systems, telecommunication networks and computer systems, etc. Many queue theoreticians have given performance analysis of retrial queues in different frame works. Reliability analysis of the retrial queue with server breakdowns was performed by [13]. [4] described the analysis of M/G/1 retrial queue with Bernoulli-schedules and general retrial times. [19] made the numerical calculation of the stationary distribution of multi-server retrial queue. The single server retrial queue with batch arrivals was provided by [12]. [9] considered M/G/1 retrial queue with general retrial times. Analysis of a multi-server retrial queue with search of customers from the orbit was made by [23]. [18] derived the busy period of the M/G/1 queue with finite retrial group. An M/G/1 retrial G-queue with preemptive resume and feedback under N-policy vacation subject to the server breakdowns and repairs was investigated by [27].

The queueing systems in which arrivals occur in batches are referred to as bulk arrival queueing system. Queueing systems with batch arrivals are common in a number of real situations such as computer and communication systems; messages which are to be transmitted could consist of a random number of packets. [15] studied a bulk input queueing system with different vacations. The bulk arrival on retrial policy concept has also attracted many researchers working in the area of queueing theory [22] considered a batch arrival Poisson queue with N policy $M^X/G/1$ retrial queue with multiple vacations and starting failures was investigated by Krishna [5]. Single server retrial queueing model according to batch Markovian arrival process with general service time was developed by [3]. [16] obtained maximum entropy solutions for batch arrival queue with an unreliable server and delaying vacations. A queue with compound Poisson arrivals, phase type required service times in which a single processor serves according to the processor-sharing discipline was investigated by [11]. [7] discussed the steady-state behavior of an $M^X/G/1$ retrial queue with an additional second phase of optional service and service interruption where breakdowns occur randomly at any instant while the server is serving the customers.

Setup or change over time is the time required by the server/machine while manufacturing one product type to switch over to another product type. The setup time generally includes times required for fixturing, tool changing and preparing the work place. [24] considered priority queues with semi-exhaustive service and class dependent setup times. [25] developed a polling model is a queueing model where many job classes share a single server and a setup time is incurred whenever the server changes class. [6] considered unrelated parallel machine scheduling with setup times using simulated annealing. A vacation queue with setup and close down times and batch Markovian arrival processes was studied by [28]. [1] provided equal processing and equal setup time cases of scheduling parallel machines with a single server. M/G/1 superposed queueing system with setup time under N-policy was studied by [20]. [2] described relevant issues in semi conductor manufacturing.

As exhibited in practical situations, repair stations or repair facilities are also subject to random breakdowns and require repair to resume its assigned jobs. The failure of any machine may cause a sequence of failures, and so a breakdown of the server may require several stages of repair. Unreliable server queueing systems are of interest from the viewpoint of practical applications in real world problems. In this investigation, we develop a retrial queueing model where the server may experience breakdowns, and broken down server requires a finite random number of stages of repair before service is restored. Various authors have analyzed queueing problems of server breakdowns with several combinations. [26] considered manufacturing systems of m identical unreliable machines producing one type of product. [21] investigated a retrial queue where server is subject to breakdown. [14] described an M/G/1 queue with second optional service and server breakdown. M/G/1 system under NT policies with breakdowns, startup and closedown was analyzed by [17]. An M/G/1 retrial queue with active breakdowns and Bernoulli schedule was investigated by [10]. [8] considered an $M^X/G/1$ with an additional second phase of optional service and unreliable server, which consist of a breakdown period and a delay period under N-policy.

Supplementary variable technique is a powerful and elegant technique to provide better solutions of non-Markovian queueing systems. In the supplementary variable technique a non Markovian process in continuous time is made Markovian process by the inclusion of one or more supplementary variables. It is often used in tackling queueing problems for steady state case.

In this investigation, we study unreliable server queueing model with bulk arrivals, retrial, setup time and two-phase repairs. By using the supplementary variable method, we obtain the steady state results for both queueing and reliability measures of interest. Further more, our aim is to illustrate graphically the effect of the various system parameters on the steady state performance measures. The remaining study is structured as follows. Next section II provides the description of model by stating requisite assumptions. Section III presents notations used for mathematical formulation of the queueing model. Section IV contains the steady state equations in terms of supplementary variables corresponding to elapsed service time, elapsed set up time and elapsed repair times of two phases. Section V is concerned with the analysis where we obtain probability distribution of the system state using the generating function technique. Some performance measures are established with the help of queue size distribution in section VI. Section VII facilitates the average total cost function involving various cost elements and some more performance measures. Some special cases, which match with the earlier existing results, are deduced in section VIII. Reliability indices of the unreliable server are obtained in section IX. Section X is devoted to the numerical illustration where sensitivity analysis is also facilitated. Finally, in section XI we conclude the study by stating the noble features and future scope of the model investigated.

II. MODEL DESCRIPTION

Consider a single server retrial queue in which customers arrive in batches according to a Poisson process. The batch size X is a random variable and $P(X=k) = C_k$, $k=1,2,3,\ldots$ with $\sum_{k=1}^{\infty} C_k = 1$. Upon arrival, the customers examine the availability of the server. If an incoming batch finds the server idle, the service of one member of the batch immediately begins and the rest of the customers in that batch join the retrial group and seek for service individually after a random amount of time. The service time provided by the single server is an independent and identically distributed random variable. The lifetime of the server is considered to be exponentially distributed. If the customers of the incoming batch notice that the server is unavailable on finding it busy, under set up or broken down state, they depart from the service area and join retrial orbit and retry for their demand after some random interval of time. Whenever the server is broken down, immediately it is sent for repair to the repairman who needs set up time before starting the first phase of repair. We assume that repair completes in two phases. After completing the first phase of repair, the server becomes available for service with probability p. However if server is not restored, it goes to second phase of repair with probability q =1-p. When both phases of repair are completed, the server becomes available immediately for service. If the server is busy at arrival epoch, then all the customers join the orbit with probability q where as if the server is free, then one of the arriving customers starts taking the service with probability p and other customers join the orbit. The setup time and repair time of two phases are independent and identically general distributed. All the considered variables are assumed to be mutually independent. To illustrate this concept we consider an example of soft drink bottling plant.

Soft drink bottling plant

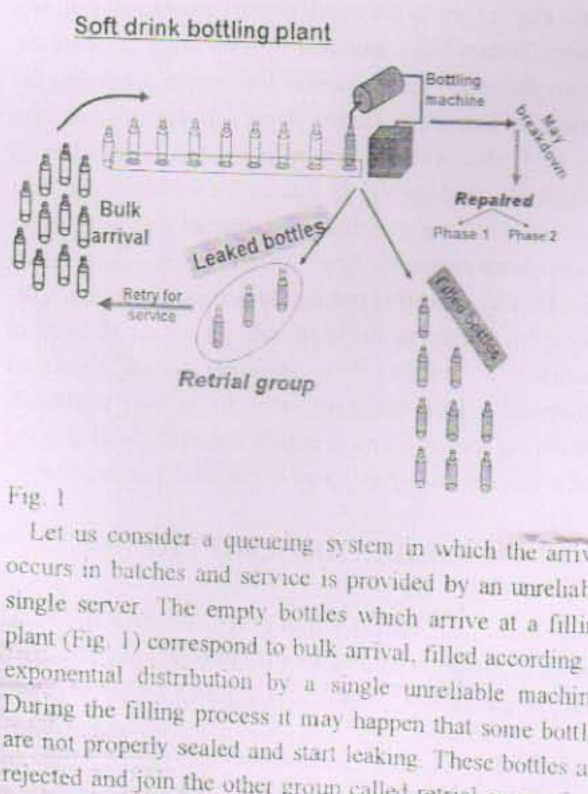


Fig. 1

Let us consider a queueing system in which the arrival occurs in batches and service is provided by an unreliable single server. The empty bottles which arrive at a filling plant (Fig. 1) correspond to bulk arrival, filled according to exponential distribution by a single unreliable machine. During the filling process it may happen that some bottles are not properly sealed and start leaking. These bottles are rejected and join the other group called retrial group, from where they retry for there service. The automatic filling plant is based on the machining system which is always prone to breakdown. If the machine breakdowns, it is repaired by the repairman who takes some setup time before starting the repair. The machine is repaired in two phases. Phase-I correspond to overall machine inspection. If there is some minor fault, it is immediately removed and machine is restored for further working. And if some major fault is found, machine is repaired in second phase which may require some more time.

III. NOTATIONS

For the mathematical formulation of $M^X/G/1$ queue, we introduce the following notations:

	: The batch arrival rate
	Mean failure rate of the server
b	Mean batch size
b_2	2^{nd} factorial moment of batch size
$C(z)$	Probability generating function of the batch size X
θ	Retrial rate
$X(t)$	Random variable denoting the elapsed service time at time t.
$Y(t)$	Random variable denoting the elapsed setup time and repair time of 1^{st} and 2^{nd} phases, respectively at time t
$b(x), a(y), h_i(y)$	Probability density function for service time, setup time and i^{th} (i=1,2) phase repair time, respectively
$B(x), A(y), H_i(y)$	: Probability distribution functions for service time, setup time and i^{th} (i=1,2) phase repair time, respectively.
$\bar{B}(x), \bar{A}(y), \bar{H}_i(y)$	: $1-B(x)$, $1-A(y)$, $1-H_i(y)$, (i=1,2), respectively.
$b^*(s), a^*(s), h_i^*(s)$	Laplace-Stieltjes transform of $b(x)$, $a(y)$, $h_i(y)$, (i=1,2), respectively.
$\mu(x), v(y), \beta_i(y)$	: Repair rate, setup rate and completion rate of i^{th} (i=1,2) phase repair, respectively
$\gamma_k, \xi_k, \eta_k^{(i)}$	k^{th} moment of service time, setup time and i^{th} (i=1,2) phase repair time distributions respectively.
$P_n(t)$	Probability that there are n jobs in retrial queue at time t when the server is in idle state.
$W_n(t,x)dx$	Joint probability that there are n jobs in the retrial queue at time t when the server is in working state and elapsed service time lies in (x, x+dx)

$S_n(t,x,y)dy$: Joint probability that there are n jobs in the retrial queue at time t when the server is in setup state and elapsed service time is x and elapsed setup time lies in (y, y+dy)

$R_{n,1}(t,x,y)dy$: Joint probability that there are n jobs in the retrial queue at time t when the server is in 1[th] phase repair state and elapsed service time is x, elapsed 1[st] phase repair time lies in (y, y+dy)

$R_{n,2}(t,x,y)dy$: Joint probability that there are n jobs in the retrial queue at time t when the server is in 2[nd] phase repair state and elapsed service time is x, elapsed 2[nd] phase repair time lies in (y, y+dy)

$L_q(z), L_s(z)$: Steady state probability generating function of the number of jobs in the retrial queue and in the system, respectively.

$R(t)$: Reliability of the server at time t

$R^*(s)$: Laplace transform of R(t)

$P_n^*(s), W_n^*(s,x), W_n^*(s,0)$: Laplace transform of $P_n(t), W_n(t,x)$ and $W_n(t,0)$, respectively

MTTF : Mean time to server failure

P(I),P(B),P(S) : Probability of server being in idle state, busy state and setup state, respectively

$P(R_1), P(R_2)$: Probability of server being in 1[st] phase repair state and 2[nd] phase repair state, respectively.

E[I],E[B],E[S] : Expected length of idle period, busy period and set up period, respectively

$E[R_j]$: Expected length of j[th] (j=1,2) phase repair period of the server

E[C] : Expected length of the cycle

We present the following hazard rates or instantaneous rates for various states of the server as follows:

- For service state

$$\mu(x)=\frac{b(x)}{\bar{B}(x)} \Rightarrow b(x)=\mu(x)\exp\left\{-\int_0^x \mu(t)dt\right\}$$

- For setup state

$$v(y)=\frac{a(y)}{\bar{A}(y)} \Rightarrow a(y)=v(y)\exp\left\{-\int_0^y v(t)dt\right\}$$

- For first phase and second phase repair states

$$\beta_i(y)=\frac{h(y)}{\bar{H}(y)} \Rightarrow h(y)=\beta_i(y)\exp\left\{-\int_0^y \beta_i(t)dt\right\},\quad i=1,2$$

k[th] moments for the service time, setup time and two phase repair time distributions are determined as

$$\gamma_k=(-1)^k b^{*(k)}(0), \xi_k=(-1)^k a^{*(k)}(0), \mathcal{R}_{k,i}=(-1)^k h_i^{*(k)}(0), i=1,2$$

IV. THE GOVERNING EQUATIONS

By supplementary variable technique, we obtain the following partial differential equations that govern the dynamics of the server's status, namely idle, busy, retrial, setup and two phases repair states as follows:

$$\left(\frac{\partial}{\partial t}+\lambda+n\theta\right)P_n(t)=\int_0^\infty \mu(x)W_n(t,x)dx \qquad ...(1)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\lambda+\alpha+\mu(x)\right)W_n(t,x)=\int_0^\infty p\beta_1(y)R_{n,1}(t,x,y)dy+\int_0^\infty q\beta_2(y)R_{n,2}(t,x,y)dy$$
$$+\lambda\sum_{k=1}^{n} C_k W_{n-k}(t,x) \qquad ...(2)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\lambda+v(y)\right)S_n(t,x,y)=\lambda\sum_{k=1}^{n} C_k S_{n-k}(t,x,y) \qquad ...(3)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\lambda+\beta_1(y)\right)R_{n,1}(t,x,y)=\lambda\sum_{k=1}^{n} C_k R_{n-k,1}(t,x,y) \qquad ...(4)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\lambda+\beta_2(y)\right)R_{n,2}(t,x,y)=\lambda\sum_{k=1}^{n} C_k R_{n-k,2}(t,x,y) \qquad ...(5)$$

The boundary conditions for the transient state are as follows:

$$S_n(t,x,0)=\alpha W_n(t,x), \qquad n\ge 0 \qquad ...(6)$$

$$W_n(t,0)=\lambda\sum_{k=1}^{n+1} C_k P_{n+1-k}(t)+(n+1)\theta P_{n+1}(t), n\ge 0 \qquad ...(7)$$

$$R_{n,1}(t,x,0)=\int_0^\infty v(y)S_n(t,x,y)dy, n\ge 0 \qquad ...(8)$$

$$R_{n,2}(t,x,0)=q\int_0^\infty \beta_1(y)R_{n,1}(t,x,y)dy, n\ge 0 \qquad ...(9)$$

The normalizing condition for the transient state is given by

$$\sum_{n=0}^\infty\left[P_n(t)+\int_0^\infty W_n(t,x)dx+\int_0^\infty\int_0^\infty S_n(t,x,y)dxdy+\int_0^\infty\int_0^\infty R_{n,1}(t,x,y)dxdy\right.$$
$$\left.+\int_0^\infty\int_0^\infty R_{n,2}(t,x,y)dxdy\right]=1 \qquad ...(10)$$

For steady state system, we define the probabilities as follows:

$$P_n=\lim_{t\to\infty}P_n(t),\quad W_n(x)=\lim_{t\to\infty}W_n(t,x),\quad S_n(x,y)=\lim_{t\to\infty}S_n(t,x,y)$$

$$R_{n,1}(x,y)=\lim_{t\to\infty}R_{n,1}(t,x,y),\quad R_{n,2}(x,y)=\lim_{t\to\infty}R_{n,2}(t,x,y)$$

Letting $t\to\infty$ in equations (1)-(10), we get the differential equations for the steady state system as follows:

$$(\lambda + n\theta)P_n = \int_0^\infty \mu(x) W_n(x)\,dx \quad \dots(11)$$

$$\left(\frac{d}{dx} + \lambda + \alpha + \mu(x)\right) W_n(x) = \int_0^\infty p\beta_1(y) R_{n,1}(x,y)\,dy + \int_0^\infty q\beta_2(y) R_{n,2}(x,y)\,dy$$

$$+ \lambda \sum_{k=1}^{n} C_k W_{n-k}(x) \quad \dots(12)$$

$$\left(\frac{d}{dy} + \lambda + \nu(y)\right) S_n(x,y) = \lambda \sum_{k=1}^{n} C_k S_{n-k}(x,y) \quad \dots(13)$$

$$\left(\frac{d}{dy} + \lambda + \beta_1(y)\right) R_{n,1}(x,y) = \lambda \sum_{k=1}^{n} C_k R_{n-k,1}(x,y) \quad \dots(14)$$

$$\left(\frac{d}{dy} + \lambda + \beta_2(y)\right) R_{n,2}(x,y) = \lambda \sum_{k=1}^{n} C_k R_{n-k,2}(x,y) \quad \dots(15)$$

The steady state boundary conditions (6)-(9) yield

$$S_n(x,0) = \alpha W_n(x) \quad \dots(16)$$

$$W_n(0) = \lambda \sum_{k=1}^{n+1} C_k P_{n+1-k} + (n+1)\theta P_{n+1} \quad (17)$$

$$R_{n,1}(x,0) = \int_0^\infty \nu(y) S_n(x,y)\,dy \quad \dots(18)$$

$$R_{n,2}(x,0) = q\int_0^\infty \beta_1(y) R_{n,1}(x,y)\,dy \quad \dots(19)$$

For the steady state, the normalizing condition (10) is given by

$$\sum_{n=0}^{\infty} P_n + \int_0^\infty W_n(x)\,dx + \int_0^\infty \int_0^\infty S_n(x,y)\,dx\,dy + \int_0^\infty \int_0^\infty R_{n,1}(x,y)\,dx\,dy$$

$$+ \int_0^\infty \int_0^\infty R_{n,2}(x,y)\,dx\,dy = 1 \quad \dots(20)$$

V. QUEUE SIZE DISTRIBUTION

For the analytic solution to resolve the system of equations (11)-(19), we introduce the following generating functions:

$$P(z) = \sum_{n=0}^{\infty} P_n z^n \,;\; W(z,x) = \sum_{n=0}^{\infty} W_n(x) z^n \,;$$

$$S(z,x,y) = \sum_{n=0}^{\infty} S_n(x,y) z^n$$

$$R_i(z,x,y) = \sum_{n=0}^{\infty} R_{n,i}(x,y) z^n,\; i = 1,2 \,;$$

$$C(z) = \sum_{k=1}^{\infty} C_k z^k, \; |z| \le 1$$

Multiplying both sides of the equation (11)-(19) by appropriate powers of z and summing, we get

$$\lambda P(z) + z\theta P'(z) = \int_0^\infty \mu(x) W(z,x)\,dx \quad (21)$$

$$\left(\frac{\partial}{\partial x} + \lambda(1 - C(z)) + \alpha + \mu(x)\right) W_n(z,x) = \int_0^\infty p\beta_1(y) R_1(z,x,y)\,dy$$

$$+ \int_0^\infty q\beta_2(y) R_2(z,x,y)\,dy \quad \dots(22)$$

$$\left(\frac{\partial}{\partial y} + \lambda(1 - C(z)) + \nu(y)\right) S(z,x,y) = 0 \quad \dots(23)$$

$$\left(\frac{\partial}{\partial y} + \lambda(1 - C(z)) + \beta_1(y)\right) R_1(z,x,y) = 0 \quad \dots(24)$$

$$\left(\frac{\partial}{\partial y} + \lambda(1 - C(z)) + \beta_2(y)\right) R_2(z,x,y) = 0 \quad \dots(25)$$

$$S(z,x,0) = \alpha W(z,x) \quad \dots(26)$$

$$W(z,0) = \lambda \frac{P(z)C(z)}{z} + \theta P'(z) \quad \dots(27)$$

$$R_1(z,x,0) = \int_0^\infty \nu(y) S(z,x,y)\,dy \quad \dots(28)$$

$$R_2(z,x,0) = q\int_0^\infty \beta_1(y) R_1(z,x,y)\,dy \quad \dots(29)$$

To obtain main results for queue size distribution, we establish following propositions using the system of equations (11)-(19) in terms of the generating functions.

Proposition-1: The partial probability generating function when the server is in idle state, busy state, setup state, first phase repair state and second phase repair state, respectively are

(i) $$P(z) = (1 - \rho\Omega_1)\exp\left[-\frac{\lambda}{\theta}\int_z^1 C(t)\frac{b^*\{Q(t)\} - t}{t\left[t - b^*\{Q(t)\}\right]}dt\right] \quad \dots(30)$$

(ii) $$W(z,x) = \frac{\lambda(C(z) - 1)}{\left[z - b^*\{Q(z)\}\right]} P(z)\exp\left[\{-Q(z)\}x\right]\bar{B}(x) \quad (31)$$

(iii) $$S(z,x,y) = \frac{\alpha\lambda(C(z) - 1)}{\left[z - b^*\{Q(z)\}\right]} P(z)\exp\left[\{-Q(z)\}x\right]\exp\left[\{-\lambda(1 - C(z))\}y\right]\bar{B}(x)\bar{A}(y)$$

$$\dots(32)$$

(iv) $$R_1(z,x,y) = \frac{\nu(y)\alpha\lambda(C(z) - 1)}{\left[z - b^*\{Q(z)\}\right]} P(z)\exp\left[\{-Q(z)\}x\right]\exp\left[\{-2\lambda(1 - C(z))\}y\right]$$

$$\times \bar{B}(x)\bar{A}(y)\bar{H}_1(y) \quad (33)$$

(v) $$R_2(z,x,y) = \frac{q\beta_1(y)\nu(y)\alpha\lambda(C(z) - 1)}{\left[z - b^*\{Q(z)\}\right]} P(z)\exp\left[\{-Q(z)\}x\right]$$

$$\times \exp\left[\{-3\lambda(1 - C(z))\}y\right] \times \bar{B}(x)\bar{A}(y)\bar{H}_1(y)\bar{H}_2(y) \quad (34)$$

Proof: The proof can be found in the Appendix-A.

Proposition-2: The marginal generating functions of the orbit size when the server is in busy state, setup state, first phase repair state and second phase repair state, respectively are given as:

(i) $$W(z)=\frac{\lambda(C(z)-1)}{\left[z-b^*\{Q(z)\}\right]}\frac{\left[1-b^*\{Q(z)\}\right]}{Q(z)}P(z) \quad \ldots(35)$$

(ii) $$S(z)=-\frac{\alpha\,\overline{a^*}(\lambda-\lambda C(z))\,\overline{b^*}\{Q(z)\}}{\left[z-b^*\{Q(z)\}\right]\,Q(z)}P(z) \quad \ldots(36)$$

(iii) $$R_1(z)=-\frac{\alpha\overline{h_1^*}(\lambda-\lambda C(z))a^*(\lambda-\lambda C(z))}{\left[z-b^*\{Q(z)\}\right]}\frac{\overline{b^*}\{Q(z)\}}{Q(z)}P(z) \quad \ldots(37)$$

(iv) $$R_2(z)=-\frac{\alpha q\overline{h_2^*}(\lambda-\lambda C(z))h_1^*(\lambda-\lambda C(z))a^*(\lambda-\lambda C(z))}{\left[z-b^*\{Q(z)\}\right]}\times\frac{\overline{b^*}\{Q(z)\}}{Q(z)}P(z) \quad \ldots(38)$$

Proof: The results given in equations (35)-(38) are obtained using

$$W(z)=\int_0^\infty W(z,x)dx\ ,\ S(z)=\int_0^\infty\int_0^\infty S(z,x,y)dxdy$$

$$R_1(z)=\int_0^\infty\int_0^\infty R_1(z,x,y)dxdy\ ,\ R_2(z)=\int_0^\infty\int_0^\infty R_2(z,x,y)dxdy$$

Theorem-1:

(i) The probability generating function of the orbit size is

$$L_o(z)=\frac{(z-1)(1-\rho\Omega_1)}{\left[z-b^*\{Q(z)\}\right]}\exp\left[-\frac{\lambda}{\theta}\int_z^1\frac{C(t)b^*\{Q(t)\}-t}{t\left[t-b^*\{Q(t)\}\right]}\right] \quad \ldots(39)$$

(ii) The probability generating function of the system size is

$$L_s(z)=\frac{(z-1)b^*\{Q(z)\}}{\left[z-b^*\{Q(z)\}\right]}\exp\left[-\frac{\lambda}{\theta}\int_z^1\frac{C(t)b^*\{Q(t)\}-t}{t\left[t-b^*\{Q(t)\}\right]}\right] \quad \ldots(40)$$

Proof: Equations (39) and (40) can be determined by using the following results:

$$L_o(z)=P(z)+W(z)+S(z)+R_1(z)+R_2(z),$$
$$L_s(z)=P(z)+zW(z)+zS(z)+zR_1(z)+zR_2(z)$$

Theorem-2:

(i) The marginal generating function of the orbit size when the server is available

$$P(z)+W(z)+S(z)=P(z)\left[1+\frac{\lambda(C(z)-1)\left[1-b^*\{Q(z)\}\right]-\alpha\overline{a^*}(\lambda-\lambda C(z))\overline{b^*}\{Q(z)\}}{\left[z-b^*\{Q(z)\}\right]Q(z)}\right]$$

(41)

(ii) The marginal generating function of the system size when the server is available

$$P(z)+zW(z)+zS(z)=P(z)\left[1+\frac{\lambda z(C(z)-1)\left[1-b^*\{Q(z)\}\right]-\alpha z\overline{a^*}(\lambda-\lambda C(z))\overline{b^*}\{Q(z)\}}{\left[z-b^*\{Q(z)\}\right]Q(z)}\right] \quad \ldots(42)$$

Proof: The proof of the theorem is straight forward

Theorem-3:

Mean queue length of orbit and mean queue length of system size are

$$E(L_q)=\frac{\lambda}{(1-\rho\Omega_1)}\frac{1}{\theta}\left\{(b-1)+\rho\Omega_1\right\}+\frac{1}{2}\left\{\lambda\gamma_2b_2\Omega_1^2+\gamma_1(b_2\Omega_1+\alpha b_2\Omega_2)\right\}$$

...(43)

$$E(L_s)=\frac{\lambda}{(1-\rho\Omega_1)}\frac{1}{\theta}\left\{(b-1)+\rho\Omega_1\right\}+\frac{1}{2}\left[\lambda\gamma_2b_2\Omega_1^2+\gamma_1(b_2\Omega_1+\alpha b_2\Omega_2)\right]+\rho\Omega_1 \quad \ldots(44)$$

where $\rho=\lambda b\gamma_1$, $C'(1)=b$, $C''(1)=b_2$

$$\Omega_1=\left\{1+\alpha\left(\xi_1+\eta_{1,1}+q\eta_{1,2}\right)\right\},$$

$$\Omega_2=\xi_2+\eta_{2,1}+q\eta_{2,2}+2\xi_1\eta_{1,1}+2\xi_1\eta_{1,2}+2\eta_{1,1}\eta_{1,2}\}$$

Proof: Differentiating (39) and the taking limit $z-1$, we obtain the mean queue length of the orbit as given in (43). Again, differentiating (40) and the taking limit $z\to 1$, we obtain the mean queue length of the system as given in (44).

VI. PERFORMANCE MEASURES

Some performance measures are derived using propositions and theorems established in previous section.

- The probability of idle period
$$P(I)=\lim_{z\to 1}P(z) = 1-\rho\Omega_1 \quad \ldots(45)$$
- The probability of busy period
$$P(B)=\lim_{z\to 1}\int_0^\infty W(z,x)dx=\rho \quad \ldots(46)$$
- The probability of setup period
$$P(S)=\lim_{z\to 1}\int_0^\infty\int_0^\infty S(z,x,y)dxdy=\alpha\rho\xi_1 \quad \ldots(47)$$
- The probability of first phase repair period
$$P(R_1)=\lim_{z\to 1}\int_0^\infty\int_0^\infty R_1(z,x,y)dxdy=\alpha\rho\eta_{1,1} \quad (48)$$
- The probability of second phase repair period
$$P(R_2)=\lim_{z\to 1}\int_0^\infty\int_0^\infty R_2(z,x,y)dxdy=\alpha\rho\eta_{1,2} \quad \ldots(49)$$

VII. COST ANALYSIS

For any queueing system, the cost analysis constitutes an important aspect of the investigation from implementation viewpoint.

The expected length of cycle per unit time is given by

$$E(C) = E(I) + E(B) + E(S) + E(R_1) + E(R_2) \qquad ...(50)$$

where

$$P(I)=\frac{E(I)}{E(C)},\ P(B)=\frac{E(B)}{E(C)},\ P(S)=\frac{E(S)}{E(C)},\ P(R_j)=\frac{E(R_j)}{E(C)},\ j=1,2 \qquad ...(51)$$

E(B) can be obtained as (cf. Takagi, 1991)

$$E(B) = \frac{b}{1-\rho\Omega_1} \qquad ...(52)$$

Now E(C) is determined as

$$E(C) = \frac{E(B)}{P(B)} = \frac{b}{\rho(1-\rho\Omega_1)} = \frac{1}{\lambda\gamma_1(1-\rho\Omega_1)} \qquad ...(53)$$

A queueing system comprises of various cost factors. Here, in order to construct the average total cost function, the following cost elements are taken into consideration:

- C_u = Start up cost per unit time
- C_i = Cost incurred per unit time when the server is in idle state
- C_o = Cost per unit time for keeping the server on and in operation
- C_S = Setup cost per cycle
- C_{h_1} = Holding cost per unit time, which is incurred on each customer present in the orbit
- C_{h_2} = Holding cost per unit time, which is incurred on each customer present in the system
- C_d = Breakdown cost per unit time for a failed server
- C_{r_i} = Cost incurred per unit time when the server is under i^{th} (i=1,2) phase repair

The expected total cost per unit time can be expressed as

$$E(TC) = (C_u + C_d)\times\frac{1}{E(C)} + C_{h_1}E(L_1) + C_{h_2}E(L_2) + C_iP(I) + C_oP(B) + C_sP(S) + C_{r_1}P(R_1) + C_{r_2}P(R_2) \qquad ...(54)$$

VIII. SPECIAL CASES

In this investigation $M^X/G/1$ model with unreliable server subject to breakdown, setup before repair, and two phase repair states have been considered for some special cases which are deduced by setting appropriate parameters as follows:

Case I: M/G/1 model with unreliable server

In this case, we consider the single arrival of jobs Now by setting $b = 1$ and $b_2 = 0$ in equations (43) and (44), the mean queue length of the orbit and system respectively are as follows:

$$E(L_1) = \frac{\lambda^2\gamma_1\Omega_1}{\theta(1-\lambda\gamma_1\Omega_1)} \qquad (55)$$

$$E(L_2) = E(L_1) + \lambda\gamma_1 \qquad (56)$$

Case II: $M^X/G/1$ model with reliable server

In this case $\alpha = 0$ so that equations (43) and (44) convert to

$$E(L_1) = \frac{\lambda}{(1-\rho)}\left[\frac{1}{\theta}\{(b-1)+\rho\} + \frac{1}{2}b_2\lambda\gamma_2 + \gamma_1(1+\alpha\Omega_2)\right] \qquad ...(57)$$

$$E(L_2) = E(L_1) + \rho$$

$$\text{where, } \Omega_2 = \xi_2 + \eta_{2,1} + q\eta_{2,2} + 2\xi_1\eta_{1,1} + 2\xi_1\eta_{1,2} + 2\eta_{1,1}\eta_{1,2} \qquad ...(58)$$

Case III: M/G/1 model with reliable server

If there is single arrival and server is not subject to breakdown, then by putting $b = 1$, $b_2 = 0$ and $\alpha = 0$, equations (43) and (44) become

$$E(L_1) = \frac{\lambda^2\gamma_1}{\theta(1-\lambda\gamma_1)} \qquad ...(59)$$

$$E(L_2) = E(L_1) + \lambda\gamma_1 \qquad ...(60)$$

Case IV: $M^X/G/1$ model without set up time and no phase repair

For this case, we obtain

$$E(L_1) = \frac{\lambda}{1-\rho(1+\alpha\eta_{1,1})}\left[\frac{1}{\theta}\{(b-1)+\rho(1+\alpha\eta_{1,1})\} + \frac{1}{2}b_2(1+\alpha\eta_{1,1})\lambda\gamma_2(1+\alpha\eta_{1,1}) + \gamma_1\right] \qquad ...(61)$$

$$E(L_2) = E(L_1) + \rho(1+\alpha\,\eta_{1,1}) \qquad ...(62)$$

which coincide with the results determined by Wang et al (2001).

IX. RELIABILITY ANALYSIS

To determine reliability indices, we consider setup and breakdown states as absorbing states. Utilizing notations and assumptions as already presented in section 3, the transient differential difference equations corresponding to various states are as follows:

$$\left(\frac{\partial}{\partial t} + \lambda + n\theta\right)P_n(t) = \int_0^\infty \mu(x)W_n(t,x)\,dx \qquad (63)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + \alpha + \mu(x)\right)W_n(t,x) = \lambda\sum_{k=1}^{n} C_k W_{n-k}(t,x) \qquad ...(64)$$

$$W_n(t,0) = \lambda\sum_{k=1}^{n+1} C_k P_{n+1-k}(t) + (n+1)\theta P_{n+1}(t) \qquad (65)$$

with the initial condition $P_0(0) = 1$.

The differential difference equations in terms of Laplace transform governing the model are:

$$sP_n^*(s) - 1 = -(\lambda + n\theta)P_n^*(s) + \int_0^\infty \mu(x)W_n^*(s,x)\,dx \qquad (66)$$

$$sW_n^*(s,x) + \frac{\partial W_n^*(s,x)}{\partial x} = -(\lambda + \alpha + \mu(x))W_n^*(s,x) + \lambda\sum_{k=1}^{n} C_k W_{n-k}^*(s,x) \qquad (67)$$

$$W_n^*(s,0) = (n+1)\theta P_{n+1}^*(s) + \lambda \sum_{k=1}^{n+1} C_k P_{n+1-k}^*(s) \qquad (68)$$

Multiplying both sides of equations (66) to (68) by appropriate powers of z, and then summing over n=1, 2, 3,.... we obtain

$$(s+\lambda)P^*(s,z) - 1 = -z\theta\frac{\partial P^*(s,z)}{\partial z} + \int_0^\infty \mu(x)W^*(s,z,x)\,dx \qquad ...(69)$$

$$\frac{\partial W^*(s,z,x)}{\partial x} + \left[\{s+\{\lambda+\alpha+\mu(x)\}-\lambda C(z)\right]W^*(s,z,x) = 0 \qquad ...(70)$$

$$W^*(s,z,0) = \theta\frac{\partial P^*(s,z)}{\partial z} + \frac{\lambda C(z)P^*(s,z)}{z} \qquad ...(71)$$

Proposition-3:

(i) Probability generating function in the form of Laplace parameter (s) when server is in idle state is given by

$$P^*(s,z) = \int_z^w \frac{1}{\theta[b^*\{s+\alpha+\lambda(1-C(y))-y\}]}$$

$$\times\exp\left\{\frac{1}{\theta}\int_y^z \frac{x(s+\alpha)-\lambda C(x)b^*\{s+\alpha+\lambda(1-C(x))\}}{x[b^*\{s+\alpha+\lambda(1-C(x))\}-x]}dx\right\}dy \qquad ...(72)$$

where ω is the root of the equation $z = b^*\{s+\alpha+\lambda(1-C(z))\}$ and $z \neq 0$.

For $z=\omega$, $P^*(s,\omega) = \dfrac{1}{\{s+\alpha+\lambda(1-C(\omega))\}}$ ...(73)

(ii) Probability generating function in the form of Laplace parameter (s) when server is in busy state is given by

$$W^*(s,z) = \left[\frac{1}{[z-b^*\{s+\alpha+\lambda(1-C(z))\}]} + \frac{\lambda(C(z)-1)-s}{[z-b^*\{s+\alpha+\lambda(1-C(z))\}]}P^*(s,z)\right]$$

$$\times\frac{\bar{b}^*\{s+\alpha+\lambda(1-C(z))\}}{\{s+\alpha+\lambda(1-C(z))\}} \qquad ...(74)$$

(iii) Reliability of the server in term of Laplace parameter (s) is given by

$$R^*(s,z) = \left(\frac{\alpha}{s+\alpha} + \frac{1}{s+\alpha}\right)\int_1^w \frac{1}{\theta[b^*\{s+\alpha+\lambda(1-C(y))-y\}]}$$

$$\times\exp\left\{\frac{1}{\theta}\int_y^z \frac{x(s+\alpha)-\lambda C(x)b^*\{s+\alpha+\lambda(1-C(x))\}}{x[b^*\{s+\alpha+\lambda(1-C(x))\}-x]}dx\right\}dy \qquad ...(75)$$

Proof: For proof see appendix B

Theorem 4: Mean time to system failure of the server is

$$MTTF = \frac{1}{\alpha}\int_1^\infty \frac{1}{\theta[b^*\{s+\alpha+\lambda(1-C(y))-y\}]}$$

$$\times\exp\left\{\frac{1}{\theta}\int_y^1 \frac{x\lambda-\lambda C(x)b^*\{\alpha+\lambda(1-C(x))\}}{x[b^*\{\alpha+\lambda(1-C(x))\}-x]}dx\right\}dy \qquad ...(76)$$

Proof: $MTTF = \int_0^\infty R(t)dt = [R^*(s)]_{s=0}$

Using above relation, we can get the result.

Theorem 5: The steady state availability (A) and failure frequency (W_f) are given by

$$A = 1-\rho\alpha\left(\xi_1+\eta_{1,1}+q\eta_{1,2}\right) \qquad ...(77)$$

And $W_f = \alpha\rho$ ...(78)

Proof: Instantaneous availability A(t) at time t of a system is the probability that the system is operational at time t. The limiting steady state availability is defined as

$$A = \lim_{t\to\infty} A(t)$$

$$= \sum_{n=0}^\infty P_n + \sum_{n=0}^\infty \int_0^\infty W_n(x)dx = \lim_{Z\to 1}\left[P(z) + \int_0^\infty W(z,x)dx\right]$$

Using equations (30) and (46), we get the result given in (77).

The failure frequency of the server is given by

$$W_f = \sum_{n=0}^\infty \int_0^\infty \alpha W_n(x)dx = \lim\int_0^\infty \alpha W(z,x)dx$$

Using equation (46), we get the result given in (78).

X. NUMERICAL RESULTS

In this section, numerical results for the queue lengths in the orbit and in the system are calculated using MATLAB software. The graphical presentation has been done in figs 2- 7. For numerical purpose, we use the following distributions:

- Service time is k-Erlang distributed with k=2
- Batch size distribution of the arrival is geometric with mean 2
- Setup time is exponential distributed
- Repair time is exponential distributed

The effect of arrival of batch of size (b), retrial rate ( ) and server's joining probability (p) on the average number of customers in orbit $E(L_1)$ and in the system $E(L_2)$ by varying the arrival rate ( ) of batch size, mean failure rate ( ) of the server, and first moment of setup time ($_1$), is demonstrated in figures 2- 7. It is observed that both $E(L_1)$ and $E(L_2)$ increase with the increase in , and . In figures 2(a) and 3(a), we illustrate that the batch size (b) does not affect much the average number of customers in orbit and in the system for smaller values of . However for larger value of , both $E(L_1)$ and $E(L_2)$ increase moderately with b (for b=1, b=2) and for b=3 the increment is significant. In figs 2(b) and 3(b), we see how the average number of customers in the orbit and in the system change with the retrial rate ( ). It is clear that $E(L_1)$ and $E(L_2)$ decrease with the increase in retrial rate but effect is significant only for larger values of . Figures 2(c) and 3(c) depict that $E(L_1)$ and $E(L_2)$ are almost same with the increasing values of joining probability (p) and arrival rate ( ).

Figs 4(a) and 5(a) display the effect of failure rate ( ) on the both queue length $E(L_1)$ and $E(L_2)$ by varying the batch size (b). We see from the graphs that as increases, $E(L_1)$ and $E(L_2)$ remain constant for small values of b whereas for higher value of b, both queue lengths $E(L_1)$ and $E(L_2)$ increase sharply. In figs 4(b) and 5(b) we observe that as increases, queue lengths $E(L_1)$ and $E(L_2)$ increase but as increases, the decreasing trends are noticed for both the queue lengths. In figs 4(c) and 5(c), we illustrate the effect of joining probability p on both queue lengths $E(L_1)$ and $E(L_2)$. It is noted that in both cases queue lengths $E(L_1)$ and $E(L_2)$ decrease slightly when increasing values are taken for p for lower values of , but as increases, this decrease is important. However $E(L_1)$ and $E(L_2)$ increase significantly with .

The effect of batch size b on both queue lengths $E(L_1)$ and $E(L_2)$ by varying setup time ($_1$) are displayed in figs 6(a) and 7(a). We see that $E(L_1)$ and $E(L_2)$ increase with the increasing values of b and $_1$. Figs 6(b) and 7(b) illustrate the effect of retrial rate on the queue lengths. The decreasing trend in both queue lengths $E(L_1)$ and $E(L_2)$ with the increase in retrial rate is found. Figs 6(c) and 7(c) depict the effect of joining probability p on $E(L_1)$ and $E(L_2)$. It is easily observed from the graphs that in case of p=.1 and p=.5, joining probability has almost no effect on the queue lengths $E(L_1)$ and $E(L_2)$ with increasing , rather increases, these are slightly scattered. However for p=.9, the increase in queue lengths is distinguishable, with increasing . But the queue lengths $E(L_1)$ and $E(L_2)$ are found to increase as p increases.

From the graphs, overall we conclude that

- Both queue lengths $E(L_1)$ and $E(L_2)$ increase as the arrival rate ( ) and batch size (b) increase.
- With the increase in retrial rate, both queue lengths decrease.
- Both queue lengths decrease as the joining probability increases.

XI. CONCLUSION

In this investigation, we have incorporated the concepts of retrial, setup time, breakdown and two phase general repair while predicting the performance measures of $M^X/G/1$ queueing system. We have examined the effect of various parameters namely the retrial rate, arrival rate, setup rate, joining probability, etc. by taking the numerical illustration. Retrial queueing models are often used for the performance prediction of unreliable server systems such as manufacturing systems, computer systems and communication networks, etc.. The reliability and availability studies provided, can play important role in improving the performability of these systems. The proposed methodology may be helpful to analyze industrial congestion problems specifically a wide range of production/manufacturing scenarios. The system performance measures, which have been displayed graphically to show the effect of various parameters on them, can be successfully utilized to upgrade the concerned system during development and design phase. A cost function identified by costs elements involved in the system may be helpful to achieve desired goal keeping in mind the techno-economic constraints.

Future research may be conducted on the same lines by generalizing the results to k-phases repair problems or by taking the concept of bulk service.

APPENDIX- I

Proof of proposition 1:-

By using equations (25) & (29), we get

$$R_2(z,x,y) = q\beta_1(y)R_1(z,x,y)\exp\{-\lambda(1-C(z))\}y]\overline{H}_2(y) \quad ...(I.1)$$

Proceeding in the similar manner, the equations (24) and (23) can be determined by using equations (28) and (26) respectively, as

$$R_1(z,x,y) = v(y)S(z,x,y)\exp\{-\lambda(1-C(z))\}y]\overline{H}_1(y) \quad ...(I.2)$$

$$S(z,x,y) = \alpha W(z,x)\exp\{-\lambda(1-C(z))\}y]\overline{A}(y) \quad ...(I.3)$$

From equation (I.1) and with the help of equations (I.2) and (I.3), we get

$$R_2(z,x,y) = q\beta_1(y)v(y)\alpha W(z,x)\exp[\{-3\lambda(1-C(z))\}y] \overline{A}(y)\overline{H}_1(y)\overline{H}_2(y) \quad ...(I.4)$$

Similarly equation (I.2) can be written as

$$R_1(z,x,y) = v(y)\alpha W(z,x)S(z,x,y)\exp[\{-2\lambda(1-C(z))\}y]\overline{A}(y)\overline{H}_1(y) \quad ...(I.5)$$

Utilizing equations (I.4) and (I.5) in equation (22), we get

$$\left[\frac{\partial}{\partial x} + \lambda(1-C(z)) + \alpha + \mu(x)\right]W(z,x) = \int_0^\infty\int_0^\infty p\beta_1(y)v(y)\alpha W(z,x)\exp[\{-2\lambda(1-C(z))\}y]$$
$$\times \overline{A}(y)\overline{H}_1(y)dxdy + \int_0^\infty\int_0^\infty \beta_2(y)q\beta_1(y)v(y)\alpha$$
$$\times W(z,x)\exp[\{-3\lambda(1-C(z))\}y]\overline{A}(y)\times\overline{H}_1(y)\overline{H}_2(y)dxdy$$

Simplifying the above expression, we obtain

$$\left[\frac{\partial}{\partial x} + \lambda(1-C(z)) + \alpha(1-\phi(z)) + \mu(x)\right]W(z,x) = 0 \quad (I.6)$$

Here $\phi(z) = a^*(\lambda-\lambda C(z))b_1^*(\lambda-\lambda C(z))\{p+qb_2^*(\lambda-\lambda C(z))\}$

Using equation (I.6) in equation (27), we get

$$W(z,x) = W(z,0)\exp[-Q(z)]x\,\overline{B}(x)$$

$$=\left[\frac{\lambda P(z)C(z)}{z}+\theta P'(z)\right]\exp\{-Q(z)x\}\bar{B}(x) \qquad (I.7)$$

Removing the term W(z,x) from equations (21) and (I.7), we find

$$P'(z)=\left\{\frac{\lambda}{\theta}\frac{C(z)b^{*}\{Q(z)\}-z}{z[z-b^{*}\{Q(z)\}]}\right\}P(z) \qquad ...(I.8)$$

On solving (I.8), we get

$$P(z)=c\exp\left\{-\frac{\lambda}{\theta}\int_{z}^{1}\frac{C(t)b^{*}\{Q(t)\}-t}{t[t-b^{*}\{Q(t)\}]}dt\right\} \qquad ...(I.9)$$

We compute the value of W(z,0) with the help of equations (27) and (I.8) as

$$W(z,0)=\frac{\lambda(C(z)-1)}{z-b^{*}\{Q(z)\}}P(z) \qquad ...(I.10)$$

Now we determine of P(1), W(1,0), W(1,x), S(1,x,y), $R_1(1,x,y)$ and $R_2(1,x,y)$. Taking limit z → 1 in equations (I.9), (I.10), (I.7)

$$P(1)=c \qquad ...(I.11)$$

$$W(1,0)=\frac{\lambda bP(1)}{1-\rho\Omega_1} \qquad ...(I.12)$$

$$W(1,x)=\frac{\lambda b}{1-\rho\Omega_1}P(1)\bar{B}(x) \qquad ...(I.13)$$

$$S(1,x,y)=\frac{\alpha\lambda b}{1-\rho\Omega_1}P(1)\bar{B}(x)\bar{A}(y) \qquad ...(I.14)$$

$$R_1(1,x,y)=\frac{v(y)\alpha\lambda b}{1-\rho\Omega_1}P(1)\bar{B}(x)\bar{A}(y)\bar{H}_1(y) \qquad ...(I.15)$$

$$R_2(1,x,y)=\frac{q\beta_1(y)v(y)\alpha\lambda b}{1-\rho\Omega_1}P(1)\bar{B}(x)\bar{A}(y)\bar{H}_1(y)\bar{H}_2(y) \qquad ...(I.16)$$

Also taking the limit z → 1 in equation (20), we find from the normalizing condition

$$c=1-\rho\Omega_1 \qquad ...(I.17)$$

To find the equation (30), we put the value of c in equation (I.9). We obtain equation (31), by eliminating W(z,0) from equation (I.7) and (I.10). After using equation (31), we obtain the equations (32), (33) and (34).

APPENDIX- II

Proof of proposition 3:-

Using equations (72)-(73), we get

$$W^{*}(s,z,x)=W^{*}(s,z,0)\exp[-\{s+\alpha+\lambda(1-C(z))\}x]\bar{B}(x) \qquad (II.18)$$

$$=\left[\frac{\lambda C(z)P^{*}(s,z)}{z}+\frac{\theta\partial P^{*}(s,z)}{\partial z}\right]\exp[\{-(s+\alpha+\lambda(1-C(z)))\}x]\bar{B}(x) \qquad (II.19)$$

From equations (69) and (II.19), we find

$$\theta[z-b^{*}\{s+\alpha+\lambda(1-C(z))\}]\frac{\partial P^{*}(s,z)}{\partial z}=1+\frac{[\lambda C(z)b^{*}\{s+\alpha+\lambda(1-C(z))\}-(s+\lambda)]}{z}P^{*}(s,z) \qquad ...(II.20)$$

Here the coefficient of $F(z)=[z-b^{*}\{s+\alpha+\lambda(1-C(z))\}]$

$$F(0)=-b^{*}(s+\alpha+\lambda)<0$$

$$F(1)=1-b^{*}(s+\alpha)\geq 0$$

$$F'(z)=1-b^{*'}\{s+\alpha+\lambda(1-C(z))\}(-\lambda C'(z))$$

$$F''(z)=b^{*''}\{s+\alpha+\lambda(1-C(z))\}(-\lambda C'(z))^{2}+b^{*'}\{s+\alpha+\lambda(1-C(z))\}(\lambda C''(z))\leq 0$$

Now equation (I.20) becomes

$$\frac{\partial P^{*}(s,z)}{\partial z}=\frac{1+[\lambda C(z)b^{*}\{s+\alpha+\lambda(1-C(z))\}-(s+\lambda)]}{\theta[z-b^{*}\{s+\alpha+\lambda(1-C(z))\}]z}P^{*}(s,z)$$

or .

$$\frac{\partial P^{*}(s,z)}{\partial z}-\frac{[\lambda C(z)b^{*}\{s+\alpha+\lambda(1-C(z))\}-(s+\lambda)]}{\theta[z-b^{*}\{s+\alpha+\lambda(1-C(z))\}]z}P^{*}(s,z)$$

$$=\frac{1}{\theta[z-b^{*}\{s+\alpha+\lambda(1-C(z))\}]z} \qquad ...(II.21)$$

Solving the above equation, we get the equation (72)

Computation of $W^{*}(s,z)$

Using equation (I.19), we have

$$W^{*}(s,z)=\int_{0}^{\infty}\left[\frac{\lambda C(z)P^{*}(s,z)}{z}+\frac{\theta\partial P^{*}(s,z)}{\partial z}\right]\exp[\{-(s+\alpha+\lambda(1-C(z)))\}x]\bar{B}(x)dx \qquad ...(II.22)$$

Simplifying the above expression; we obtain the equation (74).

The value of $R^{*}(s)$

$$R^{*}(s)=P^{*}(s,1)+W^{*}(s,1)$$

$$=P^{*}(s,1)+\left[\frac{1}{1-b^{*}(s+\alpha)}+\frac{-sP^{*}(s,1)}{1-b^{*}(s+\alpha)}\right]\frac{\overline{b^{*}}(s+\alpha)}{s+\alpha}$$

$$=\left(\frac{1+\alpha}{s+\alpha}\right)P^{*}(s,1) \qquad ...(II.23)$$

After using equation (72),we get equation (75).

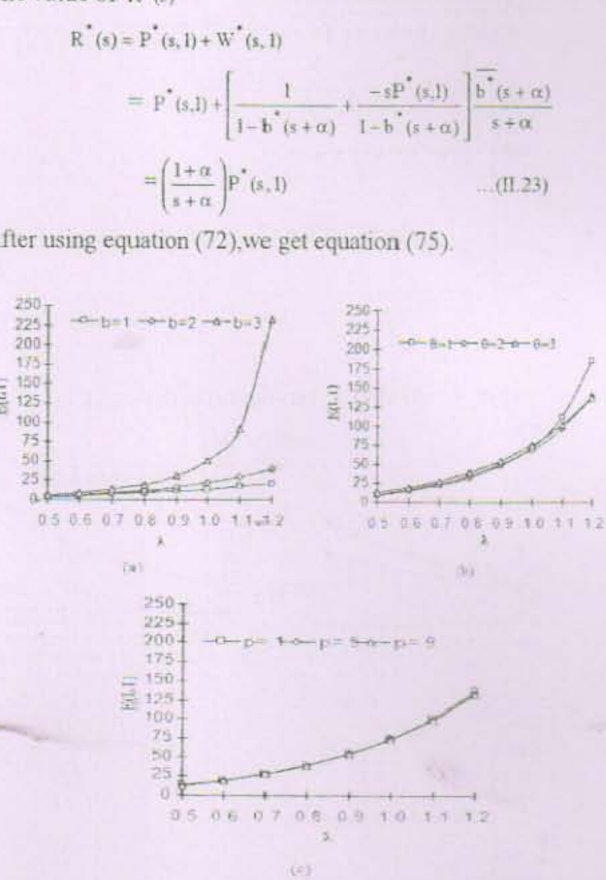


Fig. 2 : $E(L_l)$ vs □ varying (a) b, (b) θ and (c) p

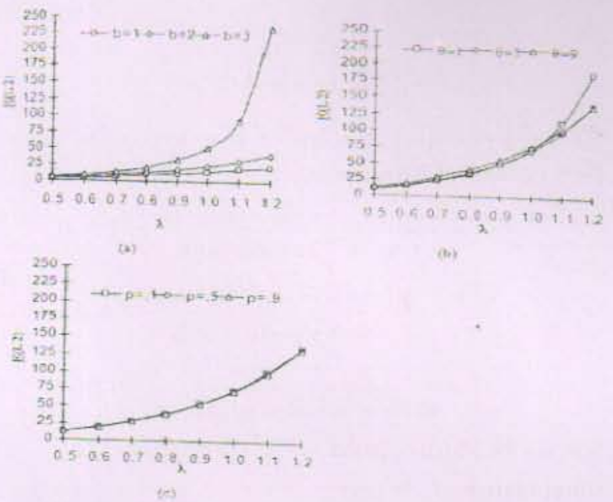


Fig. 3 : $E(L_2)$ vs varying (a) b, (b) θ and (c) p

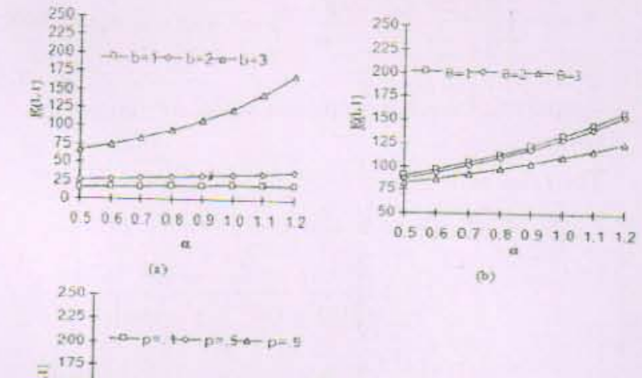


Fig 4 $E(L_1)$ vs varying (a) b, (b) θ and (c) p

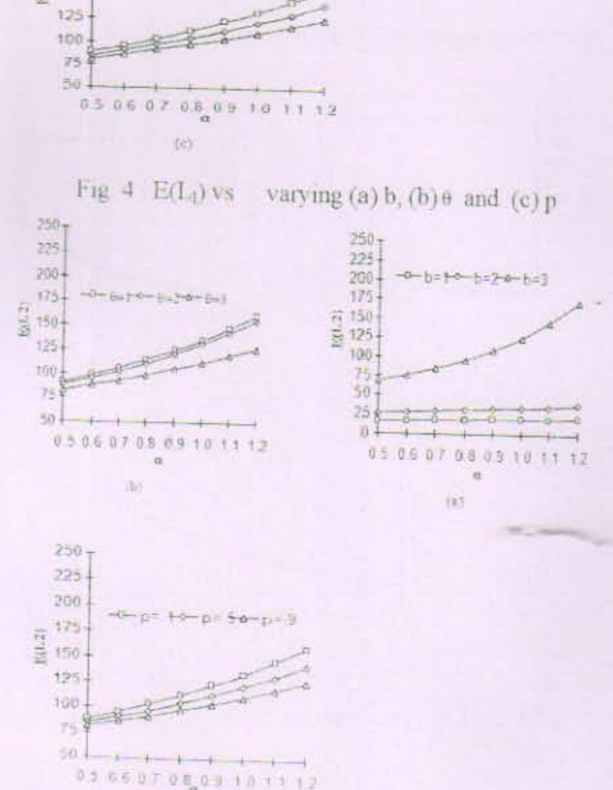


Fig 5 $E(L_2)$ vs varying (a) b, (b) θ and (c) p

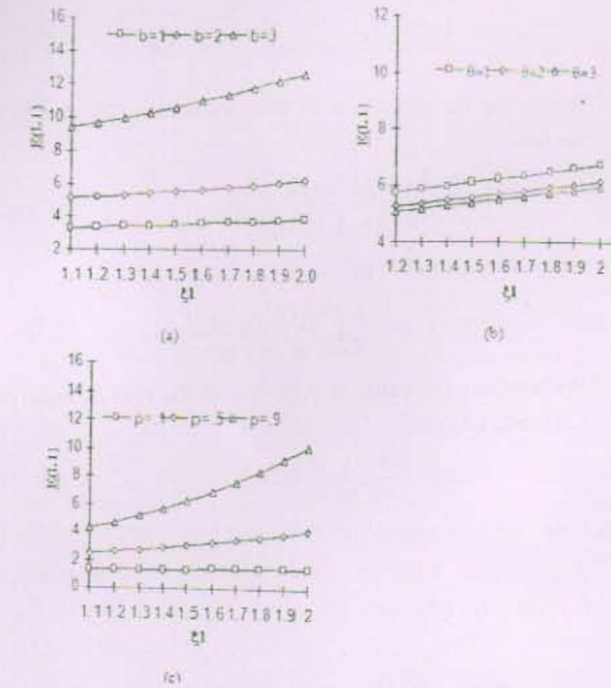
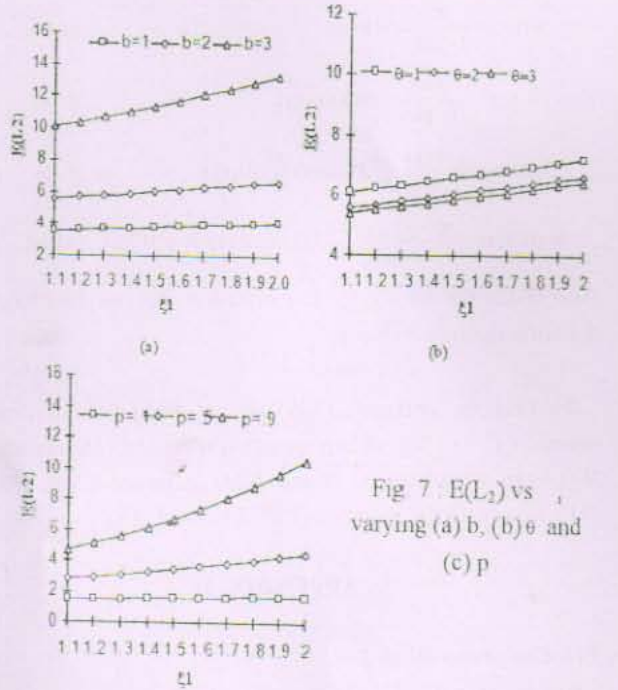


Fig 6 : $E(L_1)$vs ,varying (a) b, (b) θ and (c) p



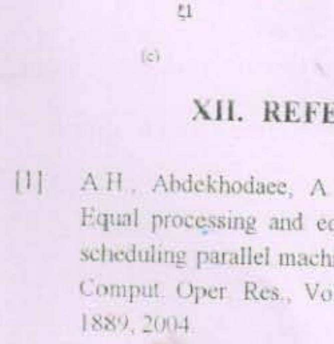


Fig 7 $E(L_2)$ vs , varying (a) b, (b) θ and (c) p

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