Fixed Point Theorem in Fuzzy 3 Metric Space

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ABSTRACT
In this paper we prove a common fixed point theorem for three mappings in fuzzy metric space, fuzzy 2 metric spaces and fuzzy 3-metric spaces. **AMS**: 54H25, 47H10.

Keywords: fixed point, fuzzy 2 metric spaces, and fuzzy 3 metric spaces.

I INTRODUCTION

In [31] the concept of fuzzy sets was introduced by Zadeh. Deng [8], Erceg [10], Kaleva and Seikkala [23], Kramosil and Michalek [25] have introduced the concept of fuzzy metric spaces in different ways. Many authors have also studied the fixed point theory in these fuzzy metric spaces are [1], [6], [11], [17], [20], [21], [22], [27] and for fuzzy mappings [2], [3], [4], [5],[19], [26]. Recently Wenzhi[30] and many others initiated the study of Probabilistic 2-metric spaces (or 2-PM spaces). We know that 2-metric space is a real valued function of a point triples on a set X, whose abstract properties were suggested by the area function in Euclidean spaces. Now it is natural to expect 3-metric space which is suggested by the volume function. The method of introducing this is naturally different from 2-metric space theory. In this paper we prove a common fixed point theorem for three mappings in fuzzy metric space, and then extend this result to fuzzy 2 metric spaces and fuzzy 3-metric spaces. This result is motivated by [32].

II PRELIMINARIES

Def.2.1. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0,1]$is called a t-norm in($[0, 1]$, $*$) if for all a, b, c, d$\in [0, 1]$following conditions are satisfied:

i. $a*1 = a$,
ii. $a*b = b*a$,
iii. $a*b \leq c*d$ whenever $a\leq c$ and $b\leq d$,
iv. $a*(b*c) = (a*b)*c$.

Def.2.2. The 3-tuple (X, M, $*$) is called a fuzzy metric space (FM space) if X is an arbitrary set, $*$ is a continuous t- norm and M is a fuzzy set in $X^2\times[0,\infty]$ satisfying the following conditions for all x, y, z $\in$ X and s, t > 0

FM-1 M(x, y, 0) = 0,
FM-2 M(x, y, t) = 1 $\forall$t>0 iff x= y,
FM-3 M(x, y, t) = M(y, x, t),
FM-4 $M(x, y, t) * M(y, z, s)\leq M (x, z, t+s)$
FM-5 M(x, y, .):$[0, 1)\rightarrow[0, 1]$ is left continuous ,
FM-6 $\lim_{t\rightarrow\infty}M(x, y, t)= 1$.

Lemma 2.1: M(x, y, z,.)is non decreasing for all for all x, y, z $\in$ X.

Def.2.3.let (X, M, $*$) is called a fuzzy metric space:

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t > 0
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if if $\lim_{n\rightarrow\infty}M(x_{n+p}, x_n, t)= 1$ $\forall$ t> 0and p > 0
3. A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 2.2: let $\{y_n\}$ be a sequence in fuzzy metric space with the condition (FM-6). If there exist a number $q \in (0,1)$ such that
$M (y_{n+2}, y_{n+1},qt) \geq M (y_{n+1}, y_n, t)$
for all t > 0and n= 1 ,2, …. , then $\{y_n\}$ is a Cauchy sequence.

Lemma 2.3: If for all x, y in X, t > 0 and for a number$q \in (0,1)$, M (x, y, qt) $\geq$M (x, y, t), then x = y

Remark: Lemma 2.1, 2.2 and 2.3 hold for fuzzy 2 metric spaces and fuzzy 3 metric spaces also.

Def.2.4: A function M is continuous in fuzzy metric space iff whenever $x_n \to x$, $y_n \to y$ then

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$$

for each $t > 0$

Def.2.5: Two mappings A and Son a fuzzy metric space Xare said toweakly commuting if
M (ASx, SAx, t) $\geq$M (Ax, Sx, t) ,$\forall$x$\in$ X and t > 0.

Def.2.6: A binary operation $*$: $[0, 1] \times [0, 1] \times [0, 1] \rightarrow [0,1]$ is called acontinuous t- norm if ([0, 1], $*$) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, for all $a_1, b_1, c_1, a_2, b_2, c_2$ in [0,1].

Def.2.7: The 3-tuple (X, M, $*$) is called a fuzzy 2-metric space (FM space) if X is an arbitrary set, $*$ is a continuous t- norm and M is a fuzzy set in

$X^3 \times [0,\infty]$ satisfying the following conditions for all x, y, z, u∈ X and $t_1, t_2, t_3 > 0$

FM'-1 M(x, y, z, 0) = 0,

FM'-2 M(x, y, z, t) = 1, ∀t > 0 iff x= y,

FM'-3 M(x, y, z, t) = M(x, z, y, t) = M(y, z,x, t) (Symmetric about three variables)

FM'-4 M(x, y, z, $t_1+t_2+t_3$) ≥ M(x, y,u, t_1) *M (x, u, z, t_2)*M (u, y, z, t_3)

(This corresponds to tetrahedron inequality in 2-metric space)

The function valueM(x, y, z,t) may be interpreted as the probability that the area of triangle is less than t.

FM'-5 M(x, y, z, .):[0, 1)→[0, 1] is left continuous ,

Def.2.8: Let (X, M, *) is called a fuzzy 2 metric space:

1. A sequence $\{x_n\}$ in fuzzy 2 metric space X is said to be convergent to a point $x \in X$if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all a in X and t > 0
2. A sequence $\{x_n\}$ in fuzzy 2 metric space X is called a Cauchy sequence if if $\text{Lim}_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ for all a in X and t> 0, p > 0
3. A fuzzy 2 metric space in which every Cauchy sequence is convergent is said to be complete.

Def.2.9: A function M is continuous in fuzzy 2 metric space iff whenever $x_n \to x$, $y_n \to y$ then $\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a, t)$ for all $a \in X$ and $t > 0$.

Def.2.10: Two mappings A and Son a fuzzy 2 metric space X are said to weakly commuting if M (ASx, SAx, a, t) ≥ M (Ax, Sx, a, t),∀x, a ∈ X and t > 0.

Def.2.11: A binary operation *: $[0, 1]^4 \to [0,1]$ is called a continuous t- norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \le a_2 * b_2 * c_2 * d_2$ whenever $a_1 \le a_2$, $b_1 \le b_2$, $c_1 \le c_2$, $d_1 \le d_2$ for all $a_1, b_1, c_1, a_2, b_2, c_2$ and d_1, d_2 are in [0,1].

Def.2.12: The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is a continuous t- norm and M is a fuzzy set in $X^4 \times [0,\infty]$ satisfying the following conditions for all x, y, z, u,w ∈ X and $t_1, t_2, t_3, t_4 > 0$

FM''-1 M(x, y, z, w, 0) = 0,

FM''-2 M(x, y, z, w, t) = 1, ∀t > 0 iff x= y, Only when three simplex ⟨x, y, z, w ⟩degenerate

FM''-3 M(x, y, z,w, t) = M(x, w, z, y, t) = M(y, z, w, x, t)= M(z, w, x, y, t) =…. (Symmetric about three variables)

FM''-4 M(x, y, z, w,$t_1+t_2+t_3+t_4$) ≥ M(x, y, z, u, t_1) *M (x, y, u, w, t_2)*M (x, u, z, w, t_3)*M (u, y, z, w, t_4)

FM''-5 M(x, y, z,w, .):[0, 1)→[0, 1] is left continuous ,

Def.2.13: let (X, M, *) is called a fuzzy 3 metric space:

1. A sequence $\{x_n\}$ in fuzzy 3 metric space X is said to be convergent to a point $x \in X$if $\lim_{n\to\infty} M(x_n, x, a, b, t) = 1$ for all a ,b in X and t > 0
2. A sequence $\{x_n\}$ in fuzzy 3 metric space X is called a Cauchy sequence if if $\text{Lim}_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1$ for all a, b in Xandt> 0, p > 0
3. A fuzzy 3 metric space in which every Cauchy sequence is convergent is said to be complete.

Def.2.14: A function M is continuous in fuzzy 3 metric space iff whenever $x_n \to x$, $y_n \to y$ then $\lim_{n\to\infty} M(x_n, y_n, a, , b, t) = M(x, y, a, b, t)$ for all $a, b \in X$ and $t > 0$.

Def.2.15: Two mappings A and Son a fuzzy 3metric space X are said to weakly commuting if

M (ASx, SAx, a, b, t) ≥ M (Ax, Sx, a, b, t) ,∀x, a, b∈ X and t > 0.

Lemma 2.3: let (X, M, *) be a fuzzy 2 metric space. If there exist k ∈ (0,1) such that M(x ,y, z, kt) ≥M (x, y, z, t) for all for all x, y, z ∈ X with z ≠ x , z ≠ y and t>0then x =y.

III MAIN RESULT

Theorem 3.1: Let(X, M, *) be a complete fuzzy metric space with the condition (FM6)&Let S &T be continuous mappings of X in X, then S & T have common fixed point in X if there exist a continuous mapping A of X into S(X)∩T(X) which commutes with S & T and

$$M(Ax, Ay, qt) \ge \min\{M(Sx, Ay, t), M(Tx, Ax, t), M(Ty, Ax, t)\}$$

(3.1.1)

for all$x, y, z \in X, t > 0$ & $0 < q < 1$.Then S , T& A have a unique common fixed point.

Proof: we define sequences $\{x_n\}$ such that $Ax_{2n} = Sx_{2n-1}$ and $Ax_{2n-1} = STx_{2n}$, n =1 ,2, ..

We shall prove that$\{Ax_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ & $y = x_{2n+1}$ in (3.1.1), we write

$$M(Ax_{2n}, Ax_{2n+1}, qt) \ge \min\left\{\begin{matrix} M(Sx_{2n}, Ax_{2n+1}, t), \\ M(Tx_{2n}, Ax_{2n}, t), \\ M(Tx_{2n+1}, Ax_{2n}, t) \end{matrix}\right\}$$

$$\ge \min\left\{\begin{matrix} M(Ax_{2n+1}, Ax_{2n+1}, t), \\ M(Ax_{2n-1}, Ax_{2n}, t), \\ M(Ax_{2n}, Ax_{2n}, t) \end{matrix}\right\}$$

$$\ge M(Ax_{2n-1}, Ax_{2n}, t)$$

$$\ge M(Ax_{2n-1}, Ax_{2n}, t/q)$$

Therefore $M(Ax_{2n}, Ax_{2n+1}, qt) \geq M(Ax_{2n-1}, Ax_{2n}, t/q)$
By induction
$M(Ax_{2k}, Ax_{2m+1}, qt) \geq M(Ax_{2k-1}, Ax_{2m}, t/q)$
For every k and m in N. further if $2m+1 > 2k$ then

$$M(Ax_{2k}, Ax_{2m+1}, qt) \geq M(Ax_{2k-1}, Ax_{2m}, t/q) \geq \ldots \geq M\left(Ax_0, Ax_{2m+1-2k}, t/q^{2k}\right) \quad (3.1.2)$$

If $2k > 2m+1$ then

$$M(Ax_{2k}, Ax_{2m+1}, qt) \geq M(Ax_{2k-1}, Ax_{2m}, t/q) \geq \ldots\ldots\ldots \geq M\left(Ax_{2k-(2m+1)}, Ax_0, t/q^{2m+1}\right) \quad (3.1.3)$$

By simple induction with (3.1.2) & (3.1.3) we have
$M(Ax_n, Ax_{n+p}, qt) \geq M(Ax_0, Ax_p, t/q^n)$
For $n = 2k$, $p = 2m+1$ & by (FM-4)

$$M(Ax_n, Ax_{n+p}, qt) \geq M(Ax_0, Ax_1, t/2q^n) * M(Ax_1, Ax_p, t/2q^n) \quad 3.1.4)$$

If $n = 2k$, $p = 2m$ or $n = 2k+1$, $p = 2m$, for every positive integer p & n in N, by noting that $M(Ax_0, Ax_p, t/q^n) \to 1$ as $n \to \infty$.
Thus$\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete, there exist $z = \lim_{n\to\infty} Ax_n$ & $z = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Tx_{2n}$
It follows that $Az = Sz = Tz$ &

$$M(Az, A^2z, qt) \geq M(Az, AAz, qt) \geq \min\{M(Sz, AAz, t), M(Tz, Az, t), M(TAz, Az, t)\}$$
$$\geq \min\{M(Sz, ATz, t), M(Az, Az, t), M(ATz, Az, t)\}$$
$$\geq \min\{M(Sz, ATz, t), M(Az, Az, t), M(ATz, Sz, t)\}$$
$$\geq M(Sz, ATz, t) \geq M(Sz, AAz, t) \geq M(Az, A^2z, t) \ldots\ldots\ldots \geq M(Az, A^2z, t/q^n)$$

Since $\lim_{n\to\infty} M(Az, A^2z, t/2q^n) = 1$ so $Az = A^2z$
Thus z is common fixed point of A, S & T.
For uniqueness, let w (w ≠ z) be another common fixed point of S, T & A. by (3.1.1) we write
$M(Az, Aw, qt) \geq \min\{M(Sz, Aw, t), M(Tz, Az, t), M(Tw, Az, t)\}$

This implies $M(z, w, qt) \geq M(z, w, t)$
Therefore by lemma 2.3, we write z = w. This completes the proof of the theorem 3.1
Now we prove theorem for fuzzy 2 metric spaces.

Theorem 3.2: Let(X, M, *) be a complete fuzzy 2-metric space & Let S &T be continuous mappings of X in X, then S & T have common fixed point in X if there exist a continuous mapping A of X into S(X)∩T(X) which commutes with S & T and

$$M(Ax, Ay, a, qt) \geq \min\left\{\begin{matrix} M(Sx, Ay, a, t), M(Tx, Ax, a, t), \\ M(Ty, Ax, a, t) \end{matrix}\right\} \quad (3.2.1)$$

for all$x, y, a \in X, t > 0$&$0 < q < 1$.

$$\lim_{t\to\infty} M(x, y, z, t) = 1 \text{ for all } x, y, z \in X. \quad (3.2.2)$$

Then S, T& A have a unique common fixed point.
Proof: we define sequences $\{x_n\}$ such that
$Ax_{2n} = Sx_{2n-1}$ and $Ax_{2n-1} = STx_{2n}$, n =1 ,2, ..
We shall prove that$\{Ax_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ & $y = x_{2n+1}$ in (3.2.1), we write

$$M(Ax_{2n}, Ax_{2n+1}, a, qt) \geq \min\left\{\begin{matrix} M(Sx_{2n}, Ax_{2n+1}, a, t), \\ M(Tx_{2n}, Ax_{2n}, a, t), \\ M(Tx_{2n+1}, Ax_{2n}, a, t) \end{matrix}\right\}$$
$$\geq \min\left\{\begin{matrix} M(Ax_{2n+1}, Ax_{2n+1}, a, t), \\ M(Ax_{2n-1}, Ax_{2n}, a, t), \\ M(Ax_{2n}, Ax_{2n}, a, t) \end{matrix}\right\}$$
$$\geq M(Ax_{2n-1}, Ax_{2n}, a, t) \geq M(Ax_{2n-1}, Ax_{2n}, a, t/q)$$

Therefore
$M(Ax_{2n}, Ax_{2n+1}, a, qt) \geq M(Ax_{2n-1}, Ax_{2n}, a, t/q)$
By induction
$M(Ax_{2k}, Ax_{2m+1}, a, qt) \geq M(Ax_{2k-1}, Ax_{2m}, a, t/q)$
For every k and m in N. Further if $2m+1 > 2k$ then

$$M(Ax_{2k}, Ax_{2m+1}, a, qt) \geq M(Ax_{2k-1}, Ax_{2m}, a, t/q) \geq \ldots \geq M\left(Ax_0, Ax_{2m+1-2k}, a, t/q^{2k}\right)$$

… 3.2.3)
If $2k > 2m+1$ then

$$M(Ax_{2k}, Ax_{2m+1}, a, qt) \geq M(Ax_{2k-1}, Ax_{2m}, a, t/q) \ldots\ldots \geq M\left(Ax_{2k-(2m+1)}, Ax_0, a, t/q^{2m+1}\right) \quad (3.2.4)$$

By simple induction with (3.2.3) & (3.2.4) we have
$M(Ax_n, Ax_{n+p}, a, qt) \geq M(Ax_0, Ax_p, a, t/q^n)$
For $n = 2k$, $p = 2m+1$ & by (FM-4)

$$M(Ax_n, Ax_{n+p}, a, qt) \geq M(Ax_0, Ax_p, Ax_1, a, t/3q^n) * M(Ax_0, Ax_1, a, t/3q^n) * M(Ax_1, Ax_p, a, t/3q^n) \quad (3.2.5)$$

If $n=2k$, $p=2m$ or $n=2k+1$, $p=2m$, for every positive integer p & n in N, by noting that $M\left(Ax_0, Ax_p, a, t/q^n\right) \to 1$ as $n \to \infty$.

Thus $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete, there exists

$z = \lim_{n\to\infty} Ax_n$ & $z = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Tx_{2n}$

It follows that $Az = Sz = Tz$ &

$$\begin{aligned}
&M\left(Az, A^2z, a, qt\right)\\
&\geq M(Az, AAz, a, qt)\\
&\geq \min\{M(Sz, AAz, a,t), M(Tz, Az, a,t), M(TAz, Az, a,t)\}\\
&\geq \min\{M(Sz, ATz, a,t), M(Az, Az, a,t), M(ATz, Az, a,t)\}\\
&\geq \min\{M(Sz, ATz, a,t), M(Az, Az, a,t), M(ATz, Sz, a,t)\}\\
&\geq M(Sz, ATz, a,t)\\
&\geq M(Sz, AAz, a,t)\\
&\geq M\left(Az, A^2z, a,t\right)\\
&\ldots\ldots\ldots\\
&\geq M\left(Az, A^2z, a, t/q^n\right)
\end{aligned}$$

Since $\lim_{n\to\infty} M\left(Az, A^2z, a, t/q^n\right) = 1$ so $Az = A^2z$

Thus z is common fixed point of A, S & T.

For uniqueness, let w (w ≠ z) be another common fixed point of S, T & A. by (3.2.1) we write

$$M(Az, Aw, a, qt) \geq \min\{M(Sz, Aw, a,t), M(Tz, Az, a,t), M(Tw, Az, a,t)\}$$

This implies $M(z,w,a,qt) \geq M(z,w,a,t)$

Therefore by lemma 2.3, we write z = w. This completes the proof of the theorem 3.2

Now we prove theorem 3.1 for fuzzy 3 metric spaces.

Theorem 3.3: Let(X, M, *) be a complete fuzzy 3-metric space & Let S &T be continuous mappings of X in X, then S & T have common fixed point in X if there exist a continuous mapping A of X into S(X)∩T(X) which commutes with S & T and

$$M(Ax, Ay, a, b, qt) \geq \min\left\{\begin{matrix} M(Sx, Ay, a, b, t), \\ M(Tx, Ax, a, b, t), \\ M(Ty, Ax, a, b, t) \end{matrix}\right\} \qquad (3.3.1)$$

for all $x, y, a, b \in X, t > 0$ & $0 < q < 1$.

$$\lim_{t\to\infty} M(x,y,z,w,t) = 1 \text{ for all } x,y,z,w \in X. \qquad (3.3.2)$$

Then S, T& A have a unique common fixed point.

Proof: we define sequences $\{x_n\}$ such that

$Ax_{2n} = Sx_{2n-1}$ and $Ax_{2n-1} = STx_{2n}$, n =1 ,2, ..

We shall prove that $\{Ax_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ & $y = x_{2n+1}$ in (3.3.1), we write

$$\begin{aligned}
M(Ax_{2n}, Ax_{2n+1}, a, b, qt) &\geq \min\left\{\begin{matrix} M(Sx_{2n}, Ax_{2n+1}, a, b, t), \\ M(Tx_{2n}, Ax_{2n}, a, b, t), \\ M(Tx_{2n+1}, Ax_{2n}, a, b, t) \end{matrix}\right\}\\
&\geq \min\left\{\begin{matrix} M(Ax_{2n+1}, Ax_{2n+1}, a, b, t), \\ M(Ax_{2n-1}, Ax_{2n}, a, b, t), \\ M(Ax_{2n}, Ax_{2n}, a, b, t) \end{matrix}\right\}\\
&\geq M(Ax_{2n-1}, Ax_{2n}, a, b, t)\\
&\geq M(Ax_{2n-1}, Ax_{2n}, a, b, t/q)
\end{aligned}$$

Therefore

$M(Ax_{2n}, Ax_{2n+1}, a, b, qt) \geq M(Ax_{2n-1}, Ax_{2n}, a, b, t/q)$

By induction

$M(Ax_{2k}, Ax_{2m+1}, a, b, qt) \geq M(Ax_{2k-1}, Ax_{2m}, a, b, t/q)$

For every k and m in N. Further if $2m+1 > 2k$ then

$$\begin{aligned}
M(Ax_{2k}, Ax_{2m+1}, a, b, qt) &\geq M(Ax_{2k-1}, Ax_{2m}, a, b, t/q)\\
&\ldots\\
&\geq M\left(Ax_0, Ax_{2m+1-2k}, a, b, t/q^{2k}\right)
\end{aligned} \qquad (3.3.3)$$

If $2k > 2m+1$ then

$$\begin{aligned}
M(Ax_{2k}, Ax_{2m+1}, a, b, qt) &\geq M(Ax_{2k-1}, Ax_{2m}, a, b, t/q)\\
&\ldots\\
&\geq M\left(Ax_{2k-(2m+1)}, Ax_0, a, b, t/q^{2m+1}\right)
\end{aligned} \qquad (3.3.4)$$

By simple induction with (3.3.3) & (3.3.4) we have

$M\left(Ax_n, Ax_{n+p}, a, b, qt\right) \geq M\left(Ax_0, Ax_p, a, b, t/q^n\right)$

For $n=2k$, $p=2m+1$ & by (FM-4)

$$\begin{aligned}
&M\left(Ax_n, Ax_{n+p}, a, b, qt\right)\\
&\geq M\left(Ax_0, Ax_p, a, Ax_1, t/4q^n\right)\\
&* M\left(Ax_0, Ax_p, Ax_1, b, t/4q^n\right)\\
&* M\left(Ax_0, Ax_1, a, b, t/4q^n\right)\\
&* M\left(Ax_1, Ax_p, a, b, t/4q^n\right)
\end{aligned} \qquad (3.3.5)$$

If $n=2k$, $p=2m$ or $n=2k+1$, $p=2m$, for every positive integer p & n in N, by noting that $M\left(Ax_0, Ax_p, a, b, t/q^n\right) \to 1$ as $n \to \infty$.

Thus $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete, there exists

$z = \lim_{n\to\infty} Ax_n$ & $z = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Tx_{2n}$

It follows that $Az = Sz = Tz$ &

$$\begin{aligned}
&M\left(Az, A^2z, a, b, qt\right)\\
&\geq M(Az, AAz, a, b, qt)\\
&\geq \min\left\{\begin{matrix} M(Sz, AAz, a, b, t), M(Tz, Az, a, b, t), \\ M(TAz, Az, a, b, t) \end{matrix}\right\}\\
&\geq \min\left\{\begin{matrix} M(Sz, ATz, a, b, t), M(Az, Az, a, b, t), \\ M(ATz, Az, a, b, t) \end{matrix}\right\}
\end{aligned}$$

$$\geq \min\begin{Bmatrix} M(Sz, ATz, a, b, t), M(Az, Az, a, b, t), \\ M(ATz, Sz, a, b, t) \end{Bmatrix}$$

$$\geq M(Sz, ATz, a, b, t)$$

$$\geq M(Sz, AAz, a, b, t)$$

$$\geq M(Az, A^2 z, a, b, t)$$

....

$$\geq M(Az, A^2 z, a, b, t/q^n)$$

Since $\lim_{n\to\infty} M(Az, A^2 z, a, b, t/q^n) = 1$ so $Az = A^2 z$

Thus z is common fixed point of A, S & T.

For uniqueness, let w (w ≠ z) be another common fixed point of S, T & A. By (3.3.1) we write

$$M(Az, Aw, a, b, qt) \geq \min\{M(Sz, Aw, a, b, t), M(Tz, Az, a, b, t), M(Tw, Az, a, b, t)\}$$

This implies $M(z, w, a, b, qt) \geq M(z, w, a, b, t)$

Therefore by lemma 2.3, we write z = w. This completes the proof of the theorem 3.3.

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