# SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATION: A NEW NUMERICAL APPROACH

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## Abstract

*Numerical analysis is used to solve those algebraic and transcendental equations, which are difficult to solve by usual Mathematical methods. Methods like Bisection, Regula falsi and Newton-Raphson are generally used for this purpose. Among these methods, Newton-Raphson is considered as best, but it fails where the function has the point of inflexion in the domain of interval. The present method removes this difficulty. The main advantage of the developed method is that there is no need of checking the initial abscissas that vanishes the derivative at which the Newton-Raphson method fails.*

**Keywords:- Numerical analysis, Algebraic equation, Transcendental equation, Conventional numerical method.**

## I. INTRODUCTION

A frequent problem in defining the theoretical concepts of applied Engineering and allied sciences is the solution of the equations in the form $f(x) = 0$ that especially contain the mixture of algebraic and transcendental functions. However, there may be problems that contains purely non-linear equation and in particular have the terms superior to quadratic order or may be problems of purely transcendental nature. Iterative methods like Bisection, Regula falsi and Newton-Raphson are often applied to obtain the approximation of such non-linear mathematical equations [1-6]. The basic assumptions underlying its evaluation are discussed in the following paragraph.

### 1.1 Bisection method

Let, $f(X) = 0$ is an equation whose solution needs to be evaluated. If $f(X)$ is a continuous in the closed interval $\{[X_o, X_a]\}$ containing the real root thereby allots opposite sign to $f(X_o)$ and $f(X_1)$ respectively, then there must exists at least one root between these points which can be approximated by their arithmetic mean. The root obtained in this way is more nearer to the real root of the equation. The procedure is repeated to get the final root at the desired level of accuracy.

### 1.2 Regula falsi method

It is the modified form of bisection method containing the same theoretical bases. This method consists of refining the part of the curve between the domain points by means of the chord joining these points and their intersection with X-axis.

### 1.3 Newton-Raphson method

This method involves the expansion of Taylor's series by neglecting the terms of second and higher order. Geometrically, the method consists of finding the intersection of tangent line with X-axis. This method works comparatively faster than other two. However, it yields undefined result at the point where the derivative of the function vanishes.

## II. THE PRESENT METHOD

The aim of the proposed method is to present an innovative concept to solve all linear and nonlinear or transcendental functional problems free from derivatives. This method remains give the defined result where the best level conventional method of Newton-Raphson fails. The method has been developed by assuming the function f (X) to be continuous and differentiable within each point of its domain and at least in the part of domain in which the real root of the equation lie.

### 2.1 Principal of the present method

Let the function defining the equation f (x) = 0 is continuous and differentiable in its domain and at least in the part of domain in which the real root of the equation lies. The strategy of basic principal is clear in figure -1. Let

L ($X_a$, f ($X_a$)) and M ($X_O$, f ($X_O$)) are two points such that f ($X_O$) and f ($X_a$) are opposite in sign and the ordinate of the given equation f (x) = 0 follows the inequality | f ($X_O$) | < | f ($X_a$) |.

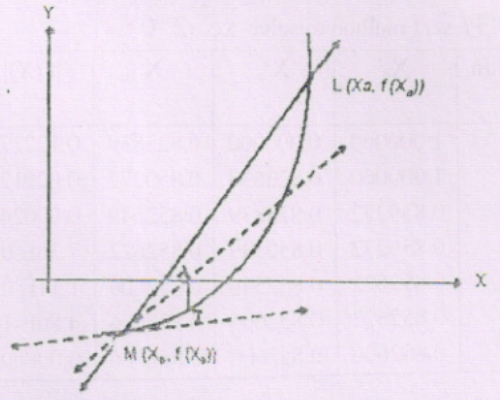


Figure 1: The geometric interpretation of proposed formula

Clearly, the equation of the line LM will be

$$f(X) - f(X_O) = m_1 ( X - X_O ) + C \qquad (1)$$

Where, C is the length of the intercept made by the line on Y – axis, and $m_1 = \frac{f(X_a) - f(X_o)}{X_a - X_o}$ represents the slope of the line.

Let, a tangent is drawn through the point M. The slope bisector of the line containing L & M and the tangent at M is obtained. The equation of this slope bisector will be

$$f(X) - f(X_o) = \frac{m_1 + f'(X_0)}{2}(X - X_0) \qquad (2)$$

$$X = X_0 + \frac{2(f(X) - f(X_o))}{m_1 + f'(X_0)} \qquad (3)$$

It passes through X - axis and meets it at the point "A". The value of abscissa correspond to this point gives the initial approximation of the root of the given equation which comes out to be,

$$X = X_0 - \frac{2f(X_o)}{m_1 + f'(X_0)} \qquad (4)$$

Substituting the value of $m_1$ we obtain,

$$X = X_0 - \frac{2f(X_0)(X_a - X_0)}{[f(X_a) - f(X_0)] + f'(X_0)(X_a - X_0)} \qquad (5)$$

Thus the projection from the point A on the curve meeting it at the point T has the Coordinates,

$$\left(X_0 - \frac{2f(X_0)(X_a - X_0)}{[f(X_a) - f(X_0)] + f'(X_0)(X_a - X_0)},\ f\left(X_0 - \frac{2f(X_0)(X_a - X_0)}{[f(X_a) - f(X_0)] + f'(X_0)(X_a - X_0)}\right)\right) \qquad (6)$$

These coordinates can be used to find the solution of the targeting type of problems. Here, the abscissa represents the iterative formula whereas; the ordinate defines the value of absolute error for the confronting problem. The method is termed as "Slope bisection method" because it is based on the bisection of the slopes of the tangent and non-tangent lines.

### 2.2 Iteration Algorithm

The process algorithm has been capsulated in Fig. 2. The basic terminologies assisting the algorithm are discussed below.

1. Choose two points $X_0$ and $X_a$ such that f ($x_0$) and f ($x_a$) are opposite in sign. Here, the nomenclature of the points ($X_0$, f ($x_0$)) and ($X_a$, f ($x_a$)) are assigned by obtaining the magnitude values of the ordinate at these points for given equation f(x) = 0 such that |f($x_0$)| <|f ($x_a$)|.
2. Get the roots of the unsolved equation from equation (6).
3. Abscissa obtained in this step defines either $X_0$ or $X_a$ of the next step. The choice of new abscissa depends on the magnitude of the distance of the new value of the root from $X_0$ and $X_a$. In particular, only the coordinate with greater distance is replaced keeping the other intact.
4. The process is repeated to get the result from next iteration, which in turn will give more exact result.
5. The process terminates when required level of accuracy is obtained.

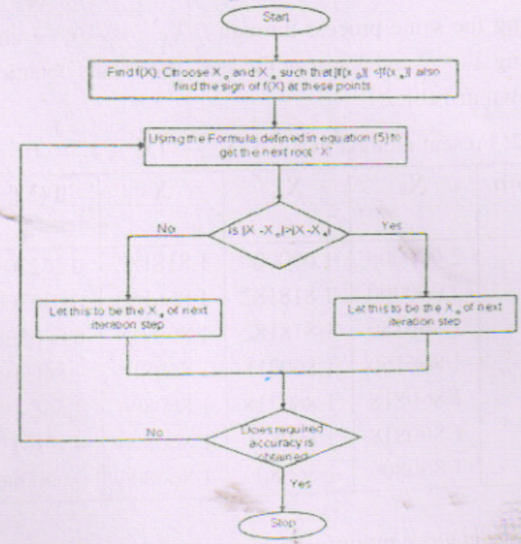


Fig.2 Flow Chart of the Processing steps

## III. NUMERICAL RESULTS AND DISCUSSIONS

In this section, the efficiency and accuracy of the method is illustrated by the following examples. This section has been divided to discuss the said two types of problems.

### *Type1– Problems containing non-transcendental terms or say purely algebraic terms*

**Example 1.** At first consider a general non-linear problem of cubic order

$$X^3 - 2X - 5 = 0 \qquad (7)$$

Here we take $f(X) = X^3 - 2X - 5$. Estimation of $f(2)$ and $f(3)$ prove that their values are opposite signed with $|f(2)| < |f(3)|$. In this regard we allot $X_0 = 2$ and $X_a = 3$ as the initial approximation to start the iteration step. The results are calculated from the defined algorithm and the different outcomes are capsulated in Table1.

**Table 1**: Present method to solve $X^3 - 2X - 5 = 0$

| Iteration No. | $X_0$ | $X_a$ | X | \|f (X)\| |
|---|---|---|---|---|
| 1 | 2.000000 | 3.000000 | 2.074074 | 0.225931 |
| 2 | 2.000000 | 2.074074 | 2.097800 | 0.036323 |
| 3 | 2.097800 | 2.074074 | 2.094536 | 0.000176 |
| 4 | 2.097800 | 2.094536 | 2.094554 | 0.000033 |
| 5 | 2.094554 | 2.094536 | 2.094551 | 0.000000 |

**Example 2.** Another general problem of same sequence is considered in equation (8).

$$X^3 - 4X + 1 = 0 \qquad (8)$$

Adopting the same process through $f(X) = x^3 - 4x + 1$ and assigning $X_0 = 2$ and $X_a = 1$ at the start, the results obtained are shown in Table2.

**Table 2**: Present method to solve $X^3 - 4X + 1 = 0$

| Iteration No. | $X_0$ | $X_a$ | X | \|f (X)\| |
|---|---|---|---|---|
| 1 | 2.000000 | 1.000000 | 1.818182 | 0.262209 |
| 2 | 2.000000 | 1.818182 | 1.866150 | 0.034300 |
| 3 | 1.866150 | 1.818182 | 1.860718 | 0.000558 |
| 4 | 1.866150 | 1.860718 | 1.860818 | 7.80E-05 |
| 5 | 1.860818 | 1.860718 | 1.860806 | 2.56E-09 |
| 6 | 1.860818 | 1.860806 | 1.860806 | 4.17E-10 |
| 7 | 1.860806 | 1.860806 | 1.860806 | 0.000000 |

### *Type2–Problems containing transcendental terms*

**Example 3.** Consider a problem incorporating exponential function

$$xe^x - 2 = 0 \qquad (9)$$

The various results for $f(X) = X e^x - 2$ and $X_0 = 1$ and $X_a = 0$ are shown in Table3.

**Table 3**: Present method to solve $xe^x - 2 = 0$

| Iteration No. | $X_0$ | $X_a$ | X | \|f (X)\| |
|---|---|---|---|---|
| 1 | 1.000000 | 0.000000 | 0.823839 | 0.122278 |
| 2 | 1.000000 | 0.823839 | 0.859272 | 0.029122 |
| 3 | 0.859272 | 0.823839 | 0.852549 | 0.000244 |
| 4 | 0.859272 | 0.852549 | 0.852622 | 7.36E-05 |
| 5 | 0.852622 | 0.852549 | 0.852606 | 1.11E-09 |
| 6 | 0.852622 | 0.852606 | 0.852606 | 4.80E-10 |
| 7 | 0.852606 | 0.852606 | 0.852606 | 0.000000 |

**Example 4.** Another type of problem of non-natural log is considered below.

$$X \log_{10} X = 1.2 \qquad (10)$$

Here, $f(X) = X \log_{10} X - 1.2$. Since $f(2)$ and $f(3)$ are opposite in sign and $|f(3)| < |f(2)|$, so we consider $X_0 = 3$ and $X_a = 2$ at the start. The iterative results are shown in Table 4.

**Table 4**: Present method to solve $X \log_{10} X = 1.2$

| Iteration No. | $X_0$ | $X_a$ | X | \|f (X)\| |
|---|---|---|---|---|
| 1 | 3.000000 | 2.000000 | 2.734175 | 0.005641 |
| 2 | 3.000000 | 2.734175 | 2.743356 | 0.002364 |
| 3 | 2.743356 | 2.734175 | 2.740646 | 4.04E-07 |
| 4 | 2.743356 | 2.740646 | 2.740646 | 2.91E-07 |
| 5 | 2.740646 | 2.740646 | 2.740646 | 1.78E-15 |
| 6 | 2.740646 | 2.740646 | 2.740646 | 4.22E-15 |
| 7 | 2.740646 | 2.740646 | 2.740646 | 0.000000 |

The correct value of the root and the absolute value of error are shown in the corresponding tables at the last two columns. Illustrated examples shows that the proposed formula has a good level of accuracy gives the result within the limited iterative steps.

### 3.1 Superiority and Advantages over others

In many ways the proposed method is superior to the generally used conventional methods. This is proved by the following discussion. Consider a quadratic equation $X^2 - 1 = 0$. Clearly, we take $f(X) = X^2 - 1$ and hence $f'(X) = 2X$. Let, we require the negative root of this problem. So, we proceed further by finding $f(-2) = 3$ & $f(0) = -1$. Clearly, we assign $X_0 = 0$.

Here Newton's formula becomes undefined at the first iteration and we cannot work further with this initial approximation. In the same case if the method of false position i.e. Regula falsi method is applied by taking the two initial abscissas to be -2 and 0. the first iteration result will be -0.5. Bisection method gives the result for same chosen values to be -1. On the other hand, starting the proposed formula by taking $X_0 = 0$ and $X_a = -2$, the first approximation gives the correct solution i.e. -1. This result shows that the proposed formula runs smoothly in those cases where Newton's methods either diverges or becomes undefined.

Revisiting the same problem by taking two new abscissas to be 0.1 and -2. Since $f(-2) = 3$ & $f(0.1) = -0.99$, so the value of $X_0$ considered for Newton-Raphson method will be 0.1 that yields the first approximation result to be 5.05. When the same initial approximations are applied on Regula falsi method the first approximation result comes to be -0.420105. The use of bisection method shows the result to be -0.95.

Inspite this, when the same result is obtained by the proposed formula by taking $X_0 = 0.1$ and $X_a = -2$, the first iteration result comes to be -1.0647. Out of these results, proposed model gives the most reliable result which is most near to its real root x = 1. However in this case the result is comparable with bisection method.

## IV. CONCLUSION

The sensitive analysis done in this paper suggests that the existing formulas to obtain the roots of the equation through Numerical methods have certain limitations. The present formula overcomes these drawbacks and gives the result comparative to above methods but with faster convergence. The major conclusions are as follows:

1. The result obtained by this formula is highly perceptible. There is no need of checking the initial abscissas that whether the derivative of the function will give well defined result or not as required in Newton-Raphson method.
2. It works uninterruptedly at the point at which Newtons Raphson method fails for example at the point where derivative becomes undefined.
3. The algorithm defining the process is very easy that can be easily coded.
4. Any equation - Algebraic, transcendental or the equations containing the terms of both can be easily solved with the same approach.

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