Conceptualizations and Issues Related to Development of Algebra

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ABSTRACT
In this paper the conceptual level in the development of algebra and related issues are discussed. The conceptual level and historical phases of development of algebra and their usage is illustrated with the help of examples. In particular, utilizing the concepts is also discussed with implications for the teaching and learning of mathematics.

Keywords: Conceptual understanding, Conceptual level, Development of Algebra.

I INTRODUCTION

The history of sciences and engineering shows that many branches of mathematics have been created in order to meet their abstract, rigorous, and expressive needs. These phenomena may be conceived as that new problems require new forms of mathematics. It also indicates that the maturity of a new discipline is characterized by the maturity of its theories denoted in rigorous and efficient mathematical means.

Algebra students may often demonstrate a certain degree of proficiency when manipulating algebraic expressions and verbalizing their behaviors. Do these abilities imply conceptual understanding? What is a reliable indicator that would provide educators with a relatively trustworthy and consistent measure to identify whether students learn algebraic concepts beyond procedures? How might teachers know when the transition from 'operational' or 'process conception' to 'structural' or 'object conception' takes place? Assessing mathematics students' conceptual understanding is critical for educators to make informed decisions when selecting curriculum, planning instruction and developing an assessment program. These decisions should be supported by strong educational foundations and be rooted in theory that provides sustained basis for understanding how mathematics students construct knowledge and how mathematics can be taught. The multiyear research study described in this paper attempts to answer these questions. This paper introduces a framework for assessing students' levels of understanding of algebra.

Understanding is a logical power manifested by abstract thought. Piaget suggested that understanding in general and in mathematics in particular is a highly complex process of abstraction. He proposed the term reflective abstraction to explain the process of developing conceptual understanding. It can be said that those who have a conceptual understanding grasp the full meaning of knowledge, and can discern, interpret, compare and contrast related ideas of the subtle distinctions among a variety of situations. Conceptual understanding in algebra can be characterized as the ability to recognize functional relationships between *known,* and *unknown, independent* and *dependent variables,* and to distinguish between and interpret different representations of the algebraic concepts. It is manifested by competency in reading, writing, and manipulating both number symbols and algebraic symbols used in formulas, expressions, equations, and inequalities. Fluency in the language of algebra demonstrated by confident use of its vocabulary and meanings and flexible operation upon its grammar rules (i.e., mathematical properties and conventions) are indicative of conceptual understanding in algebra, as well.

II ROLE OF GENERALIZATION AND ABSTRACTION IN INITIAL DEVELOPMENT

An orientation to algebraic structure requires a focus on the most general and abstract characteristics of real phenomena, beginning with children’s initial class-room encounters with such phenomena. This, in turn, requires the development of voluntary attention. And as is the case with scientific concepts and theoretical learning in general, pedagogical mediation is necessary. This is accomplished in Davydov’s program by focusing children on the theoretical characteristics of real objects, objects with which they are familiar, asking them to compare such objects with respect to their length, area, volume, or weight, and to progressively refine such comparisons until they culminate in measurement itself. Small children typically make such comparisons when receiving along with a sibling or friend, a cookie, candy bar, or a glass of juice or cola. What parent is not familiar with the small accusatory finger pointing at the other’s handout and asserting that, “his cookie is bigger” or she “has more juice”?

This exploratory research study resonates with past assumptions that students are not developing accurate concept images and concept definitions of abstract algebra concepts. Professors, especially, should find this research useful since many mathematics professors may not know what the students are actually learning or not learning in their classes. Abstract algebra has historically been a course where there exists a mismatch between what the professor assumes students are learning and what knowledge students are actually attaining. Hence, providing professors a snapshot of what students identified as key concepts (or about which they indicated confusion) would be immensely beneficial.

III METHODOLOGY

This research employed a semi-structured interview protocol with both open-ended questions and construction tasks (Patton, 2002; Taylor & Bogdan, 1984; Zazkis & Hazzan, 1999). Each interview was audio recorded and ran approximately 45-60 minutes in length in a private room to ensure confidentiality. After the interviews were complete the audio was transcribed within a week of the interview. Participants were chosen based on recently (within a year or less) being enrolled in the master's level abstract algebra course and being accepted into at least the master's level mathematics graduate program. While undergraduate students typically take abstract algebra, graduate students were chosen to provide an additional level of expertise. Three students (pseudonyms: Andrew, April, and Heather) participated in this research study. Each student had taken three lecture-based abstract algebra courses—an introductory course as an undergraduate and a yearlong sequence of two courses as a graduate student. Since the purpose of this research study is to gain insight into graduate students' perspectives of abstract algebra, one of the central foci of the interview was the creation of concept maps. These maps allowed the participants and researcher to visually understand described relationships between concepts. Novak and Cañas (2008) and Trochim (1989) largely contributed to the overall research design of this activity. First, each participant was given index cards (or post-it notes) and asked to write any important or key concepts of abstract algebra on a card (one per card). When he or she was finished with this task, the participant was asked to explain each concept. Next, participants were asked to visually represent any conceptual relationships between these topics by placing their concept cards on a sheet of poster board and drawing lines or arrows between concepts that have some type of relationship. After each participant completed a concept map, he or she was asked to explain why each line was drawn. Grounded theory was utilized when analyzing the data. Thus, the data was first collected, coded, grouped by concepts, categorized, and then the theoretical results were formulated (Charmaz, 2000). The transcribed interview responses were analyzed thematically (Charmaz, 2000; Patton, 2002; Taylor & Bogdan, 1984) focusing on the students' constructed knowledge of abstract algebra. The perceived significant concepts and connections among topics in the course were explicitly emphasized. This multi-year research study was launched to address and explore the interpretative dimensions of educational phenomena (Bums, 2000; Cohen & Manion, 1992; Merrian, 1988) associated with the assessment of levels of middle school students' understanding of linear relationship with one unknown. The internal validity of this research was achieved by data, methodological, and theory triangulation (Burns, 2000; Cohen & Manion, 1992). Data triangulation was censured by collecting data from different students at different times. Methodological triangulation was achieved through different data collection methods; a comprehensive multifaceted survey and interviews with the students were used to identify issues and themes related to developing conceptual understanding and reducing levels of abstraction. The theory triangulation was related to an epistemological and ontological justification (Merrian, 1988) of the terms representations, adaptation to abstraction level, reducing level of abstraction and conceptual understanding, which are discussed later in the paper from different perspectives. The research inquiries have been addressed through analysis of the survey and analysis of students' thinking process while they were solving problems and explaining their solutions during the interviews.

(a) Instruments and sample- The survey, designed by the researcher and described below, consisted of four interrelated parts. It combined a questionnaire and a set of problems, both related to the concept of one-two-step linear equations with one unknown, which are familiar to middle school algebra students. The actual problems were presented in three different modes: words, diagrams, and symbols. While designing and refining the survey, the researcher conducted three consecutive pilot studies to attain better clarity of the items and directions, to clean up ambiguity in sentences, to check for time of completion, and to address the problems that the students had been experienced when taking the pilot survey (McMillan & Schumacker, 1997). The pilot data were collected from three different groups of middle school algebra students in the districts which administered the final version of the instrument during next years. Each pilot provided feedback which enabled the researcher to revise the items. Scores were analyzed for adequate distribution for each item in the instrument (p.183). The instrument provided consistent results when repeated (Flower, 1993). Several experts in the field confirmed the content and face validity of the instrument. An inter-item correlation was conducted for each construct to ensure the reliability of the instrument.

(b) Description of the survey and coding system- Part I has 12 items with five optional response scale (always, often, sometimes, rarely, never). The items were aimed to collect information about students' perceptions experiences and attitudes towards different representations (words, diagrams, symbols) when they dealt with algebraic problems in general. All items were broken down into several constructs and scoring codes were clustered around students' preferred mode of representation (for example, the students' perception of pictures/diagrams, numerical symbols, algebraic symbols, multiple representations). In Part II four questions each with three choices asked the students to select the response that most closely reflects their current learning practices and most preferable/ less preferable mode of thinking (mental habits) when solving linear equations with one unknown. While the items in Part I and Part II might seem redundant they provided a basis to ensure consistency and correspondence of the students' responses. In other words, if in Part I a student indicated that he/she needs to draw a picture when solving problems, it was expected that in Part II the student would indicate that he/she is comfortable to think in pictures. Part III illustrated "structurally the same" (Dreyfus & Eisenberg, 1996, p. 268) linear relationship with one unknown posed in three different representations: as a word problem, as a diagram where the unknown number was presented as a line segment, and as an algebraic equation expressed in symbols. Students were not asked to solve the problems and generate solutions, but rather to observe and explain in writing if they recognize the *same relationship* (the sum of two numbers is 28; one number is 10, what is the other number?) presented in three different modes. The coding system for the items in Part III included 4 codes from 0 to 3. The code 0 was assigned if the students made no attempt and left it blank. The code 1 was assigned to the answers which indicated that the students did not recognize (answered 'no') the same relationship presented via three different representations. The code 2 was assigned if the students recognized the relationship presented via three different representations (answered 'yes'), but did not explicitly verbalize their thinking in a clear and coherent way. The code 3 was assigned if the students recognized the same relationship (answered 'yes') and explicitly described the relationship presented via three modes. Part rV consisted of three sets (A, B, and C) of problems that involved linear relationships with one unknown to be solved using one-two-step addition/subtraction and multiplication/division; each set consisted of three problems and the students were asked to solve each problem. For each problem set in the Part IV a coding system was created. Set A had three problems presented in words that described three different linear relationships. The coding consisted of 13 codes from 0 to 12. All possible variations were considered and specifically focused on how the problems were solved. For example, it was noted if the students used only numbers and operations (trial and error, no symbols); whether they set up numerical equations or algebraic equations, or whether they used diagrams. If the diagrams were utilized, they were coded depending on the degree of clarity (obvious, apparent or vague). Set B posed three linear relationships presented in visual form via diagrams. In each diagram numbers and an unknown were illustrated as line segments. The coding system consisted of eleven codes, from 0 to 10, and focused on how the problems presented via diagrams were solved. Each code represented a combination of several variables to document if the students set up numerical or algebraic equation, used correct or incorrect procedure, produced correct or incorrect answer. Set C contained three linear equations with one unknown represented in symbols. The coding system consisted of seven codes from 0 to 6 to account for whether the students solved the problem correctly/incorrectly using trial and error method; whether they used algebraic method i.e., steps, inverse operations. All survey problems were similar in style and level of difficulty. The surveys were analyzed to examine the relationship between students' perceptions and use of multiple representations and students' ability to recognize the same linear relationship between unknown and other elements of a problem presented in different modes: words, pictures, symbols. A constant comparison method (Glaser & Strauss, 1967) was applied to develop categories, compare each category, and look for patterns of similar and distinctive attributes. During the period of four consecutive years four tiers of data were collected from 11 schools in 6 districts, four suburban and two urban with diverse populations of students. The schools were not selected randomly but were approached by the researcher with a request for participation in the study. All the participating schools used the same mathematics curriculum which claims facilitation of reasoning skills and use of multiple representations. The schools administered the survey to all 7 and 8 algebra students *{Nmat* = 753; *Nyeari* = 176; *Nyeari* = 198; *Nyear3* = 207; *Nyear4* -172). Each of the four tiers of surveys was analyzed separately and then compared to look for common themes, trends, and tendencies. The analysis of the survey led the researcher to organize all surveys in three distinct groups to form three major categories 1, which further induced generation of a hypothesis about the indicators of students ' conceptual understanding of linear relationship with one unknown. The categorization of the surveys guided the selection of the students *(N=24)* for the interviews, eight students from each group. The selection was based on the idea of representative sample from a population, which provides the potential to be able to unitize, categorize and generalize (Glaser & Strauss, 1967; Lincoln & Guba, 1985 Individual surveys from each category were identified and the interviews with selected students were scheduled in their schools. Prior to the actual interview, the researcher met with the teachers of each student to learn about the student' ability level, performance, whether they are English Language Learners, etc. The interviews offered an important layer of evidence about the students' thinking process and the development of conceptual understanding. First, the students were asked to reflect on their survey responses, and then were presented with similar problems to gain more insight into their thinking process. During the interviews the researcher focused on by-product questions such as: How do students conceive symbolic notations as a mathematical language? To what extent and in what way students use different levels of abstraction to demonstrate different meanings of letters and symbols? Is there a relationship between the form of representation the students use (verbal, diagrammatic, symbolic) and the level of their conceptual understanding; in particular, are there signs of potential progress from a procedural to structural conception of linear relationships and its properties. Students' verbal explanations were audio-taped and written notes were collected.

IV CONCLUSION

It is interesting that in 1963, Zankov proposed a reform of school mathematics in Russsia which shares some commonality with the current US reform, in which he advocated an "approach to structuring the learning process in which the emphasis shifts to the pupil's independent intellectual activity". (Elkonin 1975, p.37) Although noting that his proposal had been implemented in several schools and some improvement had been reported, Elknonin rejected Zankov's notion, citing Vygotsky's position that the *content* of instruction was more important than the method, and arguing that one could not fix the weaknesses of a curriculum with a change in teaching methodology. A theoretical approach to mathematics was essential.

As to be expected, each of the mathematics graduate students had a differing concept image and concept definition of major abstract algebra concepts. When asked to identify these concepts, April and Heather equated the time spent in class to the importance of the concept. April stated, "I think that fields are very important because we spent a lot of time discussing the different properties of fields and the different types of fields… So I felt it was really important." Likewise, Heather repeatedly defined concept importance by how long the professor discussed it in class. Andrew, on the other hand, relied on his perceived usefulness of a certain concept to determine major concepts. When asked to describe ring theory Andrew stated:

It's like you encounter rings first from like the first time you encounter math to be like the real numbers. We actually use them in our real life and everything, so in a way like this concept of rings kind of formalizes our understanding of what everything actually means. However, despite the varying concept images associated with concept importance, there were five identified concepts that were mentioned by all three students: groups, rings, fields, Galois Theory, and isometrics with geometric applications. A complete summary of the perceived important concepts of each student is found in Figure 1. In general the mathematics graduate students had difficulty articulating their

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