Solution of Dual Integral Equations by Reducing It into an Integral Equation

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ABSTRACT

The aim of this paper is to solve a dual integral equation by changing it into an integral equation by use of mellin transform whose kernel involves Generalized Hermite Polynomial with suitable parameter. We believe that there are some more possible way to reduce such dual integral equations using different transform like those of Henkel, Fourier etc. For the sake of example we choose a dual integral equation of certain type and obtained an integral equation by use of fractional operator and mellin transform.

Keywords: Generalized Hermite Polynomial; Mellin Transform; Fractional operators; Fox-H function.

I INTRODUCTION

Dual integral equations are often encountered in different branches of mathematical physics. In the solution of certain mixed boundary value problem of mathematical physics, it is worth converting the dual integral equation into an integral equation. In the present paper, we try to solve the certain type of dual integral equations, whose kernel involves Generalized Hermite Polynomial, by converting them into integral equations. Many attempts have already been made in the past to solve such problems. The following integral representation is basic tool for our illustration.

$$\int_0^\infty k_1(x,u)\, A(u)\,du = \lambda(x);\ 0 \le x \le 1$$

(1.1) $\int_0^\infty k_2(x,u)\ A(u)\,du = \omega(x);\ x > 1$

(1.2)

k_1 & k_2 are kernels defind over x-u plane.

$$H_n^r(x,\ a,\ p) = (-1)^r\ x^{-a} e^{px^r}\ D^n\,[x^a e^{px^r}\ ]\ D = \frac{d}{dx}$$

$a, r,\ p\ \ parameter.$

II THEOREM

If f is unknown function satisfying the dual integral equation.

$$\int_0^\infty (x\,|\,y)^{a_1}\ e^{-(x|y)^r}\ H_n^r(x\,|\,y; a_1, 1)\ f(y)\frac{dy}{y} = h(x);\ 0 \le x < 1$$

(2.1)

$$\int_0^\infty (x\,|\,y)^{a_2}\ e^{-(x|y)^r}\ H_n^r(x\,|\,y; a_2, 1)\ f(y)\frac{dy}{y} = g(x); 1 \le x < \infty$$

(2.2) When h and g are prescribed function and $a_1, a_2\ and\ r$ are parameters, then f is giving by

$$f(x) = \frac{1}{r}\int_0^\infty L(x\,|\,y)\ t(y)\frac{dy}{y}$$

Where

$$L(x) = H_{2,1}^{1,0}\left(x \left| \begin{matrix} (n,1) \\ (1,1)\left(\left(1 - \frac{1}{r}\,(a_1 - n)\right), \frac{1}{r}\right) \end{matrix} \right. \right)$$

and

$t(x) = h(x), 0 \le x < 1$

$$t(x) = \frac{r\,x^{-n+a_1}}{\Gamma\left(\frac{1}{r}(a_2 - a_1)\right)}$$

$$\times \int_0^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2 - a_1) - 1\right)} v^{n - a_2 + r - 1}\, g(v)\ dv; 1 \le x < \infty$$

III MATHEMATICAL PRELIMINARY

To prove the theorems we shall use Mellin transformer and fractional integral operator.

$$f^*(s) = M\,[\,f(x); s] = \int_0^\infty f(x)x^{s-1}\ dx \qquad (3.1)$$

When $s = \sigma + i\tau$ is a complex variable.

The inverse melling transform f(x) of $f^*(s)$ is given by

$$M^{-1}[\,f^*(s)] = f(x) = \frac{1}{2\pi i}\int_{\sigma + i\infty}^{\sigma + i\infty} f^*(s)\,x^{-s}\,ds \qquad (3.2)$$

By convolution theorem for mellin transform

$$M\left[\int_0^\infty k(x\,|\,y)\ f(y)\frac{dy}{y}; s\right] = k^*(s)\ f^*(s)$$

$$\int_0^\infty k(x\,|\,y)\ f(y)\frac{dy}{y} = M^{-1}\left[\ k^*(s)\ f^*(s); s\right]$$

$$= \frac{1}{2\pi\, i}\int_L k^*(s)\ f^*(s)\,x^{-s}\ ds$$

(3.3) When L is suitable contour.

Fractional integral operator

$$\tau(\alpha;\ \beta;\ r; w(x)) = \frac{r\,x^{-r\alpha + r - \beta - 1}}{\Gamma(\alpha)}\int_0^\infty (x^r - v^r)^{\alpha - 1}\, v^\beta w(v)\,dv$$

(3.4)

$$R(\ \alpha;\ \beta;\ r; w(x)) = \frac{r\,x^\beta}{\Gamma(\alpha)}\int_x^\infty\ (v^r - x^r)^{\alpha - 1}\, v^{-\beta - r\alpha + r - 1}\, w(v)\ dv$$

(3.5)

IV SOLUTION

Now taking

$$k_i(x) = x^{a_i}\, e^{-x^r} H_n^r(x; a_i; 1),\ i = 1,2$$

Then from Erdeeyi [10] We get

$$k_i^*(s) = \frac{\Gamma s\, \Gamma\frac{1}{r}(s-n+a_i)}{r\, \Gamma s-n},\ i = 1,2$$

(4.1)

Hence by use at (3.3),(2.1) & (2.2) can be written as

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s-n+a_i\right)}{\Gamma s-n} f^*(s)\, x^{-s}\, ds$$

$$= h(x);\ 0 \le x < 1$$

(4.2)

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\, \Gamma\left(\frac{1}{r}(s-n+a_2\right)}{\Gamma s-n} f^*(s)\, x^{-s}\, ds$$

$$= g(x);\ 1 \le x < \infty$$

(4.3) Now operating a (4.2) by the operator (3.5) we get

$$\frac{rx^{\beta}}{\alpha}\cdot\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s-n+a_2)\right)}{\Gamma(s-n)} f^*(s)\, x^{-s}\, ds$$

$$\int_x^{\infty} (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

$$= \frac{rx^{\beta}}{\Gamma\alpha}\int_x^{\infty} (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

Now putting $v^r = \frac{x^r}{t}$ and simplifying we get

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\, \Gamma\left(\frac{1}{r}(s-n+a_2\right)}{\Gamma s-n}\cdot\frac{x^{-s}}{\Gamma\alpha}$$

$$\int_0^1 (1-t)^{\alpha-1}.t^{\left(\frac{\beta+s}{r}-1\right)}\, dt\ \ f^*(s)ds$$

$$= \frac{rx^{\beta}}{\Gamma\alpha}\int_x^{\infty} (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

(4.4)

$$\Rightarrow \frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\, \Gamma\left(\frac{1}{r}(s-n+a_2\right)}{\Gamma s-n}\cdot\frac{x^{-s}}{\Gamma\alpha}$$

$$\times \frac{\Gamma\left(\frac{1}{r}(\beta+S)\right)\Gamma\alpha}{\Gamma\left(\alpha+\frac{1}{r}\beta+\frac{1}{r}s\right)} f^*\ (s)\, ds$$

$$= \frac{rx^{\beta}}{\Gamma\alpha}\int_x^{\infty} (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

$$\Rightarrow \frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\, \Gamma\left(\frac{1}{r}(s-n+a_2\right)}{\Gamma s-n}\ x^{-s}$$

$$\times \frac{\Gamma\frac{1}{r}(\beta+s)}{.\Gamma\left(\alpha+\frac{1}{r}\beta+\frac{1}{r}s\right)} \times f^*(s)\, ds$$

$$= \frac{rx^{\beta}}{\Gamma\alpha}\int_x^{\infty} (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

In equation (4.4), we put $\beta = -n + a_1$ and $\alpha = \frac{1}{r}(a_2 - a_1),$ so that (4.4) Changes to

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\, \Gamma\left(\frac{1}{r}(s-n+a_2\right)}{\Gamma s-n}\ x^{-s}$$

$$\times \frac{\Gamma\frac{1}{r}(s-n+a_1)}{\Gamma\frac{a_2}{r}-\frac{a_1}{r}+\frac{1}{r}(-n+a_1)+\frac{1}{r}s} \times f^*(s)\, ds$$

$$= \frac{r\, x^{-n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)}$$

$$\times \int_x^{\infty} (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{-n-a_2+r-1} g(v)dv\ \ 1 \le x \le \infty$$

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\, \Gamma\left(\frac{1}{r}(s-n+a_1\right)}{\Gamma(s-n)}\ x^{-s}\ \ f^*(s)ds$$

$$= \frac{r\, x^{-n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)} \times$$

$$\int_x^{\infty} (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{-n-a_2+r-1} g(v)dv\ \ 1 \le x \le \infty$$

(4.5)

Now we write

$t(x) = h(x), \quad 0 \le x < 1$

and $t(x) = \dfrac{r\, x^{-n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)}$

$$\times \int_x^{\infty} (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{n-a_2+r-1} g(v)\ dv; 1 \le x < \infty$$

(4.6) Now from (4.2), (4.5), (4.6) we get

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s-n+a_1\right)}{\Gamma(s-n)}\; x^{-s}\; f^*(s)ds = t(x)$$

(4.7)

Again using (3.3), (4.1) & (4.6) becomes

$$\int_0^\infty k_1(x\,|\,y)\, f(y)\frac{dy}{y} = t(x);\; 0 \le x < \infty$$

(4.8)

When $k_1(x) = x^{a_1} e^{-x^r} H_x^r(x;\, a_1;1)$

Thus pair at dual integral equation (1.1) &(1.2) we have been reduced to single integral equation (4.8). Hence by mellin transform (4.8) can be written as –

$$k_1^*(s)\, f^*(s) = T^*(s)$$

(4.9)

Where $k_1^*(s) = \dfrac{\Gamma s\,\Gamma\left(\frac{1}{r}(s-n+a_1\right)}{\Gamma(s-n)}$

and $T^*(s)$ is the mellin transform of t(x).

Now

$$F^*(s) = L^*(s)\; T^*(s)$$

(4.10)

Where

$$L^*(s) = \frac{1}{k^*(s)}$$

$$= \frac{\Gamma\, s-n}{\Gamma(s)\,\Gamma\,\frac{1}{r}(s-n+a_1)}$$

By use of definition of H – function, we get the inverse transform L(x) at $L^*(S)$ as

$$L(x) = H_{2,1}^{1,0}\left(x \left| \begin{matrix}(n,1)\\ (1,1)\left(\left(1-\frac{1}{r}\,(a_1-n)\right),\frac{1}{r}\right)\end{matrix}\right.\right)$$

(4.11)

Taking inverse mellin transform of (4.10)

$$f(x) = \int_0^\infty\; L(x\,|\,y)\; t(y)\frac{dy}{y}$$

Hence using (4.11) we get

$$f(x) = \frac{1}{r}\int_0^\infty H_{2,1}^{1,0}\left(\frac{x}{y} \left| \begin{matrix}(n,1)\\ (1,1)\left(\left(1-\frac{1}{r}\,(a_1-n)\right),\frac{1}{r}\right)\end{matrix}\right.\right) t(y)\frac{dy}{y}$$

When t (y) is given by (4.6).

Hence proved the theorem.

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